Évariste Galois constructed a family of 3-transitive groups in 1830. In 1861-1873 Émile Mathieu discovered a series of multiply transitive groups which are now named after him, including 5-transitive groups of degrees 12 and 24.

John D. Dixon, Brian Mortimer

Organising groups by the transitivity of their actions is as old as group theory itself. The idea that highly transitive group actions are scarce is basic to the discovery and classification of finite simple groups.

Marston Conder, Vaughan Jones

The devil of algebra fights with the angel of geometry.

Hermann Weyl (cited by Vladimir Arnold)
Let $G$ be a group. One says that $G$ is *highly transitive* if $G$ admits an action on an infinite set $X$ such that for any two finite ordered subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of $X$ of the same cardinality, there exists $g \in G$ such that $g(x_i) = y_i$, $i = 1, \ldots, n$. 
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$$X^{(n)} = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

for any $n = 1, 2, \ldots$
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The countable subgroup $\text{FSym}(\mathbb{Z}) \subset \text{Sym}(\mathbb{Z})$ of finite permutations (i.e., permutations identical outside some interval) is also highly transitive on $\mathbb{Z}$. It is locally finite, that is, any finitely generated subgroup of $\text{FSym}(\mathbb{Z})$ is finite. Hence, $\text{FSym}(\mathbb{Z})$ is a torsion group, that is, it contains no free subgroup.
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The subgroup $\text{Alt}(\mathbb{Z}) \subset \text{FSym}(\mathbb{Z})$ of finite even permutations is also highly transitive on $\mathbb{Z}$. This group is simple, that is, it has no proper normal subgroup.
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$$\partial(fg) = f\partial(g) + g\partial(f);$$

assume $\partial$ is \textit{locally nilpotent} (LND, for short):

$$\forall a \in \mathcal{O}(X) \ \exists m \geq 0: \partial^{(m+1)}(a) = 0;$$
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a \circ \Phi_\partial(t) = \exp(t\partial)(a) = \sum_{k=0}^{m} \frac{t^k}{k!} \partial^{(k)}(a), \quad a \in \mathcal{O}(X).
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The flow $\Phi_\partial$ of an LND $\partial$ gives rise to a $\mathbb{G}_a$-subgroup of the automorphism group $\text{Aut}(X)$. Any regular $\mathbb{G}_a$-action on $X$ arises in this way.
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- Let $X = \mathbb{A}^2 = \text{Spec}(k[x, y])$. Then $\partial = \partial/\partial y$ is an LND with the flow of shifts

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- Take $P(x) \in \ker(\partial/\partial y) = k[x]$. Then $\partial_P := P(x)\partial/\partial y$ is an LND. It generates the flow of *shears*

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DEFINITION

The **SPECIAL AUTOMORPHISM GROUP** of $X$ is the subgroup $\text{SAut}(X) \subset \text{Aut}(X)$ generated by all the $\mathbb{G}_a$-subgroups:

$$\text{SAut}(X) = \langle H = \exp(t\partial) \mid \partial \in \text{LND}(\mathcal{O}(X)) \rangle$$

**Theorem (Arzhantsev-Flener-Kaiman-Kutzchebauch-Z13)**

Assume $\text{SAut}(X)$ acts transitively on $X$. Then $\text{SAut}(X)$ is highly transitive on $X$.

**Theorem (Borel-Knop)**

An algebraic group cannot act 3-transitively on an affine variety.
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CONJECTURE (ARZHANTSEV-KUYUMJIYAN-Z (AKZ) ’19)

If $\mathrm{SAut}(X)$ acts with an open orbit $\mathcal{O}$ then there is a finite collection $\{H_1, \ldots, H_N\}$ of $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(X)$ such that the group $G = \langle H_1, \ldots, H_N \rangle$ is highly transitive on $\mathcal{O}$. 
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DEFINITIONS

- An affine variety $X$ of dimension $n$ with an action of the $n$-torus $T = \mathbb{G}_m^n$ is called toric if $T$ acts on $X$ with an open orbit.
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- An affine variety $X$ of dimension $n$ with an action of the $n$-torus $\mathbb{T} = \mathbb{G}_m^n$ is called toric if $\mathbb{T}$ acts on $X$ with an open orbit.

- $X$ is called smooth in codimension 2 if the singular locus of $X$ has codimension $\geq 3$. 
THEOREM 1 (AKZ ’19) Let $X$ be a toric affine variety with no torus factor. If $X$ is smooth in codimension two then there exists a finite collection of root $\mathbb{G}_a$-subgroups $U_1, \ldots, U_m$ of $\text{Aut}(X)$ such that $G = \langle U_1, \ldots, U_m \rangle$ acts highly transitively on the smooth locus $X_{\text{reg}}$. 
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THEOREM 2 (AKZ ’19; ANDRIST ’19) For any $n \geq 2$ one can find three $\mathbb{G}_a$-subgroups $H_1, H_2, H_3 \subset \text{Aut}(\mathbb{A}^n)$ such that $G = \langle H_1, H_2, H_3 \rangle$ is highly transitive on $\mathbb{A}^n$. 
TWO THEOREMS

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REMARK Andrist found 3 explicit LND’s on $\mathbb{A}^n$, $n \geq 2$, which generate such $\mathbb{G}_a$-subgroups $H_1, H_2, H_3$. 
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THEOREM (DARJI–MITCHELL ’08) For any $\alpha \in \text{Sym}(\mathbb{Z}) \setminus \{\text{id}\}$ there exists $\beta \in \text{Sym}(\mathbb{Z})$ such that the subgroup $G = \langle \alpha, \beta \rangle$ is highly transitive on $\mathbb{Z}$.
If $\alpha$ has finite support, then one can take a shift $x \mapsto x + n$ for $\beta$. 
The following countable groups are highly transitive:

- The nonabelian free group $F_n$, $n \geq 2$
  (McDONOUGH '77, CAMERON '87, et al.);

- $\text{Out}(F_n) = \text{Aut}(F_n) / \text{Inn}(F_n)$
  (GARION–GLASNER '13);

- The free product $G_1 \ast G_2$ with nontrivial $G_1$ and $G_2$, except for the infinite dihedral group $\mathbb{Z} / 2 \mathbb{Z} \ast \mathbb{Z} / 2 \mathbb{Z}$
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- In particular, $\text{PSL}(2, \mathbb{Z}) = \mathbb{Z} / 2 \mathbb{Z} \ast \mathbb{Z} / 3 \mathbb{Z}$.

However, it is unknown whether $\text{PSL}(2, K)$ is highly transitive
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- however, it is unknown whether $\text{PSL}(2, K)$ is highly transitive for a countable field $K$ (HULL–OSIN ’16);
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- The nonabelian free group $F_n$, $n \geq 2$ (McDONOUGH ’77, CAMERON ’87, et al.);
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- in particular, PSL$(2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z}$. However, it is unknown whether PSL$(2, K)$ is highly transitive for a countable field $K$ (HULL–OSIN ’16);
- certain amalgams, HNN-extensions, and groups acting on trees (FIMA–MOON–STALDER ’15).
LEMMA 1 (DIXON-MORTIMER) Let $G \subset \text{Sym}(X)$, and let $1 \neq N \trianglelefteq G$ be a nontrivial normal subgroup.

(a) If $G$ is 2-transitive on $X$ then $N$ is transitive on $X$.
(b) If $G$ is highly transitive on $X$ then $N$ is.
(c) No abelian group is highly transitive.

Proof of (a): $G$ preserves the partition of $X$ into the orbits of $N$ on $X$. If this partition is nontrivial, $G$ cannot be 2-transitive. (c) follows from (a).

DEFINITION A subgroup $N \subset G$ is called subnormal if there exists a series

$$G \triangleright N_1 \triangleright N_2 \triangleright \ldots \triangleright N_k = N.$$ 

COROLLARY 1 Let $N \subset G$ be a nontrivial subnormal subgroup. If $G$ is highly transitive on $X$ then $N$ is.
COROLLARY 2 A *virtually solvable group cannot be highly transitive.*
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Proof: $G$ is virtually solvable implies $G$ has a solvable normal subgroup $N$ of finite index. This $N$ can be obtained from abelian groups using extensions. Hence $N$ has a nontrivial abelian subnormal subgroup $A$, and also $G$ has. By Lemma 1, if $G$ is highly transitive then $A$ is, in contradiction with (c) of Lemma 1.
DEFINITION A finitely generated group $G$ has \textit{polynomial growth} if the “volume” $V(r)$ of the ball of radius $r$ in $G$ centered at $e_G$ (with respect to the word metric) is at most polynomial:

$$V(r) \leq \text{cst} \cdot r^N, \ \exists N \geq 0.$$
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Recall that $V(r)$ is the number of elements of $G$ representable by reduced words of length at most $r$ in the given generators. The choice of generators is irrelevant, up to a constant factor.
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**GROMOV’S THEOREM** A finitely generated group $G$ has polynomial growth if and only if $G$ is virtually nilpotent.
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**GROMOV’S THEOREM** A finitely generated group $G$ has polynomial growth if and only if $G$ is virtually nilpotent.

From Corollary 2 we deduce

**COROLLARY 3** A finitely generated group $G$ of polynomial growth cannot be highly transitive.
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**QUESTION:** Can a highly transitive, finitely generated group be of intermediate growth?
THEOREM (TITS ALTERNATIVE, TITS ’72)
Let $H \subset \text{GL}(n, K)$ be a finitely generated (arbitrary, if $\text{char}(K) = 0$) linear group over a field $K$. Then either $H$ is virtually solvable, or $H$ contains a copy of $F_2$. In particular, $G$ has either polynomial, or exponential growth.
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DEFINITION A group $G$ satisfies (restricted) Tits alternative if any (finitely generated, respectively) subgroup $H$ of $G$ either is virtually solvable, or contains a copy of $F_2$. 

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COROLLARY 4 Let a group $G$ satisfies the Tits alternative, and let a subgroup $H \subset G$ it highly transitive. Then $H \supset F_2$. In particular, $H$ has exponential growth.
THEOREM  The Tits alternative holds for the following groups:

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Theorem The Tits alternative holds for the following groups:

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- Out($F_n$) (BESTVINA-FEIGHN-HANDEL ’00);
- the mapping class groups Out($\pi_1(R)$) of a compact Riemann surface $R$ (IVANOV and McCARTHY ’84-'85).

THEOREM  Given a projective variety $V$, consider the group Bir($V$) of birational transformations of $V$. Then Bir($V$) verifies the Tits alternative if either

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THEOREM The Tits alternative holds for the following groups:

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THEOREM Given a projective variety \( V \), consider the group \( \text{Bir}(V) \) of birational transformations of \( V \). Then \( \text{Bir}(V) \) verifies the Tits alternative if either

- \( \dim(V) = 2 \) (LAMY-CANTAT-URECH ‘01–’05–’18), or
- \( V \) is a hyperkähler variety (OGUISO ‘06).
The last theorem allows to answer Demailly’s question in these two cases:

**COROLLARY** Let $V$ be an algebraic surface or a hyperkähler projective variety. If a subgroup $G \subset \text{Bir}(V)$ is highly transitive then $G$ contains a copy of $F_2$, and so, has exponential growth.
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It is unknown whether Tits’ alternative holds for $\text{Aut}(\mathbb{A}^3)$.

**EXAMPLE (LEWIS-PERRY-STRAUB ’19)** The group $G = \langle H_1, H_2 \rangle$ generated by the $\mathbb{G}_a$-subgroups

$$H_1 = \{(x, y) \mapsto (x, y + \lambda x)\}, \quad H_2 = \{(x, y) \mapsto (x + \mu y^2, y)\}$$

is highly transitive on $\mathbb{A}^2 \setminus \{0\}$. It is easily seen that $G \supset F_2$. 
CONJECTURE  Let $X$ be an affine variety of dimension $\geq 2$ defined over an algebraically closed field of characteristic zero. Consider the group

$$G = \langle U_1, ..., U_s \rangle$$

generated by $\mathbb{G}_a$-subgroups $U_1, ..., U_s$ of $\text{Aut}(X)$. Then the Tits alternative holds for $G$. If $G$ is highly transitive, then $G$ contains a free subgroup $F_2$, and so, has exponential growth.
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We prove this conjecture in the case of toric affine varieties.
Consider:

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- a base of $M$ making $M$ an integer lattice of rank $n$;
- for any $m = (m_1, \ldots, m_n) \in M$, the Laurent monomial $\chi^m = x_1^{m_1} \cdots x_n^{m_n}$;
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- for any $m = (m_1, \ldots, m_n) \in M$, the Laurent monomial $\chi^m = x_1^{m_1} \cdots x_n^{m_n}$;
- the graded affine algebra

\[
A = \bigoplus_{m \in M \cap \sigma^\vee} \mathbb{k} \chi^m \text{ with } \chi^m \cdot \chi^{m'} = \chi^{m+m'}.
\]
Consider further:

- the toric affine variety $X = \text{Spec } A$, $\dim X = n$;
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**Remark**

$X$ is normal, and any normal toric affine variety arises in this way.
Consider further:

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**REMARK**

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Consider also the following associated objects:

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**Lemma TFAE:**

- \( \sigma^\vee \) is a pointed cone, that is, \( \sigma^\vee \) contains no line;
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**LEMMA** TFAE:

- $\sigma^\vee$ is a pointed cone, that is, $\sigma^\vee$ contains no line;
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**Lemma TFAE:**

- $\sigma^\vee$ is a pointed cone, that is, $\sigma^\vee$ contains no line;
- $\sigma$ is of full dimension, that is, $\Xi$ contains a basis of $\mathbb{N}_\mathbb{Q}$;
- $X$ has no toric factor, that is, $X$ cannot be decomposed into a product $\mathbb{G}_m \times Y$, where $Y$ is a toric variety of dimension $n - 1$. 

DEFINITIONS

A derivation $\partial \in \text{Der}(A)$ is called \textit{homogeneous} if $\partial$ respects the grading, that is, sends any graded piece to another one.
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- Any homogeneous derivation is of the form $\partial = \lambda \partial_{\rho,e}$ where $\rho \in \mathbb{N}$, $e \in \mathbb{M}$,

  $$\partial_{\rho,e}(\chi^m) = \langle \rho, m \rangle \chi^{m+e} \quad \forall m \in \mathbb{M}.$$
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  \partial_{\rho, e}(\chi^m) = \langle \rho, m \rangle \chi^{m+e} \quad \forall m \in M.
  $$

- The lattice vector $e \in M$ is called the **degree** of $\partial_{\rho, e}$. 
LEMMA (LIENDO ’10) A homogeneous derivation $\partial$ is locally nilpotent if and only if $\partial = \text{cst} \cdot \partial_{\rho_i,e}$ for a Demazure root $e$ with $\langle \rho_i, e \rangle = -1$. 

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LEMMA (ROMASKEVICH ’14)

- Let $\partial = \partial_{\rho, e}$ and $\partial' = \partial_{\rho', e'}$. Then $[\partial, \partial'] = \partial_{\hat{\rho}, \hat{e}}$ where
  \[
  \hat{e} = e + e' \quad \text{and} \quad \hat{\rho} = \langle \rho, e' \rangle \rho' - \langle \rho', e \rangle \rho \in \mathbb{N}.
  \]
- If $\hat{\rho} \neq 0$ then $\deg ([\partial, \partial']) = e + e' \in M$. 
DEFINITIONS

- The *Demazure facet* $S_i$ is the convex rational polyhedron in the affine hyperplane $\mathcal{H}_i = \{\langle \rho_i, e \rangle = -1\}$ defined by the inequalities

$$\langle \rho_j, e \rangle \geq 0 \ \forall j \neq i.$$ 

It is parallel to the $i$th facet $\{\langle \rho_i, e \rangle = 0\}$ of the cone $\sigma^\vee$. 
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- The **Demazure roots** belonging to $\rho_i$ are the lattice vectors from $S_i$.

- The $G_a$-subgroup $U_e = \exp(k \partial_{\rho_i, e})$ is called the **root subgroup** associated with a Demazure root $e \in S_i$. For $e, e' \in S_i$ the root subgroups $U_e$ and $U_{e'}$ commute.
THEOREM (Arzhantsev-Z ’20) Let \( X \) be a toric affine variety with no torus factor, and let a subgroup \( G \) of \( \text{Aut}(X) \) be generated by root subgroups \( U_1, \ldots, U_s \). Then either \( G \) is a unipotent algebraic group, or \( G \) contains a free subgroup of rank two.
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PROPOSITION 1 Consider the group $H = \langle U_1, U_2 \rangle$ generated by the root subgroups $U_i = \exp(t\partial_i)$, $i = 1, 2$, associated with two different ray generators, say, $\rho_1$ and $\rho_2$, respectively. Then either $H$ is a unipotent algebraic group, or the subgroup $\langle u_1, u_2 \rangle$ is a free group of rank two for a very general pair $(u_1, u_2) \in U_1 \times U_2$. 
With our choice of $X$, the Cox ring of $\mathcal{O}(X)$ is the polynomial ring in $k$ variables. This allows to reduce to the setting where $X = \mathbb{A}^k$ and

$$u_1 = (x_1 + sx^c N_1, x_2, \ldots, x_k), \quad u_2 = (x_1, x_2 + tx^d N_2, x_3, \ldots, x_k),$$

with $s, t \in \mathbb{k}$ and monomials $N_1, N_2 \in \mathbb{k}[x_3, \ldots, x_k]$. 
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with \( s, t \in k \) and monomials \( N_1, N_2 \in k[x_3, \ldots, x_k] \).

By the Jung-van der Kulk Theorem,

\[ \text{GL}_2(K) = \text{Aff}_2(K) \ast_C \text{Jonq}(K), \]

where

- \( K = k(s, t, x_3, \ldots, x_k); \)
- \( \text{Jonq}(K) \) is the de Jonquières triangular subgroup;
- \( C = \text{Aff}_2(K) \cap \text{Jonq}(K). \)
If \( c \geq 2 \) and \( d \geq 2 \) then \( H = U_1 \ast U_2 \) and \( \langle u_1, u_2 \rangle = F_2 \) for any pair of nonunit elements \((u_1, u_2)\).
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The latter remains true for general $(u_1, u_2)$ provided $c \geq 2, d \geq 1$. 

If $(*)_{\min\{c,d\}} = 0$ then $H$ is a unipotent algebraic group.
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If $c = d = 1$ then for a suitable $(u_1, u_2)$, the group $\langle u_1, u_2 \rangle$ surjects onto $SL_2(\mathbb{Z})$ and so, contains a free subgroup of rank two.
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Let as before
\[ G = \langle U_1, \ldots, U_s \rangle, \text{ where } U_i = \exp(t \partial_i). \]

The Lie algebra $L$ generated by the root derivations $\partial_i$, $i = 1, \ldots, s$, might contain extra root derivations.
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The Lie algebra \( L \) generated by the root derivations \( \partial_i, \) \( i = 1, \ldots, s, \) might contain extra root derivations. Let \( R_i \) be the set of Demazure roots \( e_{ij} \in S_i \) of \( X \) such that \( \partial_{\rho_i, e_{ij}} \in L, j = 1, \ldots, \#(R_i). \)
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**Proposition 2** Suppose (\( \ast \)) holds for any \( e_i \in R_i, e_j \in R_j, i \neq j \), that is,

\[ \min\{\langle \rho_i, e_j \rangle, \langle \rho_j, e_i \rangle\} = 0. \]

Then \( G \) is a unipotent algebraic group.
DEFINITION A finite sequence of root derivations

\[ \mathcal{D} = (D_1, \ldots, D_t, D_{t+1}) \] where \( D_i = \partial_{\rho_j(i), e_j(i), i} \in L_j(i) \)

forms a cycle if \( D_{t+1} = D_1 \) and

\[ \langle \rho_j(i+1), e_j(i), i \rangle > 0 \ \forall i = 1, \ldots, t. \]

If this inequality holds and \( \rho_j(t+1) = \rho_j(1) \) but possibly \( D_{t+1} \neq D_1 \), we say that \( \mathcal{D} \) is a pseudo-cycle.
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LEMMA TFAE:

- \( L \) contains no pseudo-cycle;
- \( L \) contains no cycle;
- \( L \) contains no 2-cycle;
- \( (*) \) holds \( \forall e_i \in R_i, e_j \in R_j, i \neq j. \)
Under the assumption that $L$ contains no cycle, Proposition 2 is proven by Arzhantsev, Liendo, and Stasyuk, arXiv 2019. Using the above lemma, we rewrite their proof as follows.
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Suppose $(\ast)$ holds. Consider the abelian Lie subalgebras of $L$,

$$L_i = \langle \partial_{\rho_i,e} \mid e \in R_i \rangle, \ i = 1, \ldots, k.$$
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Suppose $(\ast)$ holds. Consider the abelian Lie subalgebras of $L$,

$$L_i = \langle \partial_{\rho_i}, e \mid e \in R_i \rangle, \quad i = 1, \ldots, k.$$ 

Due to $(\ast)$, for any $i \neq j$ there is an alternative:

- **either** $\langle \rho_i, e_j \rangle = 0 = \langle \rho_j, e_i \rangle \ \forall e_i \in R_i, \ \forall e_j \in R_j$,
- **or, up to a transposition**, $\exists e_i \in R_i: \langle \rho_j, e_i \rangle > 0$, and so, $\langle \rho_i, e_j \rangle = 0 \ \forall e_j \in R_j$. 

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HIGH TRANSITIVITY IN ALGEBRA AND GEOMETRY
In the first case $[L_i, L_j] = 0$, and in the second
$0 \neq [L_i, L_j] \subset L_i$. Anyway, we have $L = \bigoplus_{i=1}^{r} L_i$ and

$$\dim(L) = \sum_{i=1}^{r} \dim(L_i) = \sum_{i=1}^{r} \text{card}(R_i) = \text{card}(R).$$
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Let $\Gamma_k$ be the directed graph on the vertices $L_1, \ldots, L_k$ with edges $[L_j, L_i]$ oriented as follows:

$[L_j \rightarrow L_i]$ iff $0 \neq [L_i, L_j] \subset L_i$.

If $[L_i, L_j] = 0$, there is no edge $[L_j, L_i]$ in $\Gamma_k$. 
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If \([L_i, L_j] = 0\), there is no edge \([L_j, L_i]\) in \(\Gamma_k\).

Due to Lemma, \(L\) contains no pseudo-cycle. This means that \(\Gamma_k\) is acyclic, that is, has no oriented cycle. Then any connected component of \(\Gamma_k\) has a sink. We may assume \(L_1\) to be a sink.
Deleting this sink $L_1$ and all the incident edges yields again an acyclic directed graph $\Gamma_{k-1}$, which in turn has a sink, which we take for $L_2$. 

We show that $\dim(L_i) < +\infty \forall i = 1, \ldots, k$; for $N \gg 1$, $ad(L_j)_{N}(L_i) = 0 \forall j \geq i$, and so, $ad(L_j)_{Nk}(L_i) = 0$. 

It follows that $L$ is nilpotent and finite-dimensional.
Deleting this sink $L_1$ and all the incident edges yields again an acyclic directed graph $\Gamma_{k-1}$, which in turn has a sink, which we take for $L_2$. Finally, we renumerate the vertices in such a way that

$$[L_i, L_1] \subset L_1, \quad i = 2, \ldots, r,$$

$$[L_i, L_2] \subset L_2, \quad i = 3, \ldots, r,$$

$$\ldots$$

$$[L_r, L_{r-1}] \subset L_{r-1}.$$
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We show that

- $\dim(L_i) < +\infty \quad \forall i = 1, \ldots, k$;
- for $N \gg 1$, $\text{ad}(L_j)^N(L_i) = 0 \quad \forall j \geq i$, and so,
- $\text{ad}(L)^{nk}(L) = 0$.

It follows that $L$ is nilpotent and finite-dimensional.