HIGH TRANSITIVITY IN ALGEBRA AND GEOMETRY

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Évariste Galois constructed a family of 3-transitive groups in 1830. In 1861-1873 Émile Mathieu discovered a series of multiply transitive groups which are now named after him, including 5-transitive groups of degrees 12 and 24.

John D. Dixon, Brian Mortimer

Organising groups by the transitivity of their actions is as old as group theory itself. The idea that highly transitive group actions are scarce is basic to the discovery and classification of finite simple groups.

Marston Conder, Vaughan Jones

The devil of algebra fights with the angel of geometry.

Hermann Weyl (cited by Vladimir Arnold)

DEFINITION

Let G be a group. One says that G is *highly transitive* if G admits an action on an infinite set X such that for any two finite ordered subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of X of the same cardinality, there exists $g \in G$ such that $g(x_i) = y_i$, $i = 1, \ldots, n$.

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In other words, G is transitive on X and also on

$$X^{(n)} = \{(x_1, \dots, x_n) \in X^n \,|\, x_i \neq x_j \text{ for } i \neq j\}$$

for any $n = 1, 2, \dots$

FIRST EXAMPLES

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- The subgroup $\mathrm{Alt}(\mathbb{Z}) \subset \mathrm{FSym}(\mathbb{Z})$ of finite even permutations is also highly transitive on $\mathbb{Z}$. This group is simple, that is, it has no proper normal subgroup.

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- a (reduced, irreducible) affine algebraic variety X over $\mathbb{k}$ of dimension $n \geq 2$;
- a *derivation* of $\mathcal{O}(X)$, that is, a $\mathbb{k}$-linear map $\partial\colon \mathcal{O}(X) \to \mathcal{O}(X)$ verifying the Leibniz rule:
$$\partial(fg) = f\partial(g) + g\partial(f);$$
- assume ∂ is *locally nilpotent* (LND, for short):
$$\forall a \in O(X)\ \ \exists m \geq 0\colon \partial^{(m+1)}(a) = 0;$$

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- the flow Φ_∂ defined via

$$a \circ \Phi_\partial(t) = \exp(t\partial)(a) = \sum_{k=0}^{m} \frac{t^k}{k!}\partial^{(k)}(a), \qquad a \in \mathcal{O}(X).$$

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$$\exp(t\partial)\colon (x, y) \mapsto (x, y + t)\,.$$

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- Take $P(x) \in \ker(\partial/\partial y) = \Bbbk[x]$. Then $\partial_P := P(x)\partial/\partial y$ is an LND. It generates the flow of *shears*

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DEFINITION

The *SPECIAL AUTOMORPHISM GROUP* of X is the subgroup $\mathrm{SAut}(X) \subset \mathrm{Aut}(X)$ generated by all the $\mathbb{G}_a$-subgroups:

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THEOREM (ARZHANTSEV-FLENNER-KALIMAN-KUTZSCHEBAUCH-Z '13)

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THEOREM (BOREL-KNOP) *An algebraic group cannot act* 3*-transitively on an affine variety.*

CONJECTURE (ARZHANTSEV-KUYUMJIYAN-Z (AKZ) '19)

If $\mathrm{SAut}(X)$ *acts with an open orbit* $\mathscr{O}$ *then there is a finite collection* $\{H_1, \ldots, H_N\}$ *of* $\mathbb{G}_a$*-subgroups of* $\mathrm{Aut}(X)$ *such that the group* $G = \langle H_1, \ldots, H_N \rangle$ *is highly transitive on* $\mathscr{O}$*.*

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DEFINITIONS

- **An affine variety X of dimension n with an action of the n-torus $\mathbb{T} = \mathbb{G}_m^n$ is called *toric* if $\mathbb{T}$ acts on X with an open orbit.**
- **X is called *smooth in codimension 2* if the singular locus of X has codimension ≥ 3.**

THEOREM 1 (AKZ '19) Let X be a toric affine variety with no torus factor. If X is smooth in codimension two then there exists a finite collection of root $\mathbb{G}_a$-subgroups $U_1, \ldots, U_m$ of $\mathrm{Aut}(X)$ such that $G = \langle U_1, \ldots, U_m \rangle$ acts highly transitively on the smooth locus X_{reg}.

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THEOREM 2 (AKZ '19; ANDRIST '19) *For any $n \geq 2$ one can find three $\mathbb{G}_a$-subgroups $H_1, H_2, H_3 \subset \mathrm{Aut}(\mathbb{A}^n)$ such that $G = \langle H_1, H_2, H_3 \rangle$ is highly transitive on $\mathbb{A}^n$.*

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REMARK Andrist found 3 explicit LND's on $\mathbb{A}^n$, $n \geq 2$, which generate such $\mathbb{G}_a$-subgroups H_1, H_2, H_3.

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THEOREM (DARJI–MITCHELL '08) *For any $\alpha \in \mathrm{Sym}(\mathbb{Z}) \setminus \{\mathrm{id}\}$ there exists $\beta \in \mathrm{Sym}(\mathbb{Z})$ such that the subgroup $G = \langle \alpha, \beta \rangle$ is highly transitive on $\mathbb{Z}$.*
If α has finite support, then one can take a shift $x \mapsto x + n$ for β.

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- in particular, PSL$(2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. However, it is unknown whether PSL$(2, K)$ is highly transitive for a countable field K (HULL–OSIN '16);

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- in particular, PSL$(2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. However, it is unknown whether PSL$(2, K)$ is highly transitive for a countable field K (HULL–OSIN '16);
- certain amalgams, HNN-extensions, and groups acting on trees (FIMA–MOON–STALDER '15).

LEMMA 1 (DIXON-MORTIMER) *Let $G \subset \mathrm{Sym}(X)$, and let $1 \neq N \trianglelefteq G$ be a nontrivial normal subgroup.*
(a) *If G is 2-transitive on X then N is transitive on X.*
(b) *If G is highly transitive on X then N is.*
(c) *No abelian group is highly transitive.*

Proof of (a): G preserves the partition of X into the orbits of N on X. If this partition is nontrivial, G cannot be 2-transitive. (c) follows from (a). $\square$

DEFINITION A subgroup $N \subset G$ is called *subnormal* if there exists a series

$$G \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \ldots \trianglerighteq N_k = N\,.$$

COROLLARY 1 *Let $N \subset G$ be a nontrivial subnormal subgroup. If G is highly transitive on X then N is.*

COROLLARY 2 *A virtually solvable group cannot be highly transitive.*

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Proof: G is virtually solvable implies G has a solvable normal subgroup N of finite index. This N can be obtained from abelian groups using extensions. Hence N has a nontrivial abelian subnormal subgroup A, and also G has. By Lemma 1, if G is highly transitive then A is, in contradiction with (c) of Lemma 1. □

DEFINITION A finitely generated group G has *polynomial growth* if the "volume" $V(r)$ of the ball of radius r in G centered at e_G (with respect to the word metric) is at most polynomial:

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From Corollary 2 we deduce

COROLLARY 3 *A finitely generated group G of polynomial growth cannot be highly transitive.*

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QUESTION: *Can a highly transitive, finitely generated group be of intermediate growth?*

THEOREM (TITS ALTERNATIVE, TITS '72)

Let $H \subset \mathrm{GL}(n, K)$ be a finitely generated (arbitrary, if char$(K) = 0$) linear group over a field K. Then either H is virtually solvable, or H contains a copy of F_2. In particular, G has either polynomial, or exponential growth.

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COROLLARY 4 *Let a group G satisfies the Tits alternative, and let a subgroup $H \subset G$ it highly transitive. Then $H \supset F_2$. In particular, H has exponential growth.*

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THEOREM *Given a projective variety* V*, consider the group* $\mathrm{Bir}(V)$ *of birational transformations of* V*. Then* $\mathrm{Bir}(V)$ *verifies the Tits alternative if either*

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- $\dim(V) = 2$ **(LAMY-CANTAT-URECH '01-'05-'18)**, *or*
- V *is a hyperkähler variety* **(OGUISO '06)**.

The last theorem allows to answer Demailly's question in these two cases:

COROLLARY *Let V be an algebraic surface or a hyperkähler projective variety. If a subgroup $G \subset \mathrm{Bir}(V)$ is highly transitive then G contains a copy of F_2, and so, has exponential growth.*

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EXAMPLE (LEWIS-PERRY-STRAUB '19) The group $G = \langle H_1, H_2 \rangle$ generated by the $\mathbb{G}_a$-subgroups

$$H_1 = \{(x, y) \mapsto (x, y + \lambda x)\}, \quad H_2 = \{(x, y) \mapsto (x + \mu y^2, y)\}$$

is highly transitive on $\mathbb{A}^2 \setminus \{0\}$. It is easily seen that $G \supset F_2$.

CONJECTURE *Let X be an affine variety of dimension ≥ 2 defined over an algebraically closed field of characteristic zero. Consider the group*

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generated by $\mathbb{G}_a$-subgroups $U_1, ...U_s$ of $\mathrm{Aut}(X)$. Then the Tits alternative holds for G. If G is highly transitive, then G contains a free subgroup F_2, and so, has exponential growth.

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We prove this conjecture in the case of toric affine varieties.

Consider:

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- the graded affine algebra

$$A = \bigoplus_{m \in M \cap \sigma^\vee} \mathbb{k}\chi^m \text{ with } \chi^m \cdot \chi^{m'} = \chi^{m+m'}.$$

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REMARK

- X is normal, and any normal toric affine variety arises in this way.

DUAL CONE

Consider also the following associated objects:

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- $\sigma^\vee$ *is a pointed cone, that is,* $\sigma^\vee$ *contains no line;*
- σ *is of full dimension, that is,* Ξ *contains a basis of* $N_{\mathbb{Q}}$*;*
- X *has no toric factor, that is,* X *cannot be decomposed into a product* $\mathbb{G}_m \times Y$*, where* Y *is a toric variety of dimension* $n-1$.

DEFINITIONS

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- The lattice vector $e \in M$ is called the *degree* of $\partial_{\rho,e}$.

LEMMA (LIENDO '10) *A homogeneous derivation ∂ is locally nilpotent if and only if $\partial = \mathrm{cst} \cdot \partial_{\rho_i, e}$ for a Demazure root e with $\langle \rho_i, e \rangle = -1$.*

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LEMMA (ROMASKEVICH '14)

- *Let $\partial = \partial_{\rho,e}$ and $\partial' = \partial_{\rho',e'}$. Then $[\partial, \partial'] = \partial_{\hat{\rho},\hat{e}}$ where*

$$\hat{e} = e + e' \quad \textit{and} \quad \hat{\rho} = \langle \rho, e' \rangle \rho' - \langle \rho', e \rangle \rho \in N\,.$$

- *If $\hat{\rho} \neq 0$ then $\deg([\partial, \partial']) = e + e' \in M$.*

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- The *Demazure facet* $\mathcal{S}_i$ is the convex rational polyhedron in the affine hyperplane $\mathcal{H}_i = \{\langle \rho_i, e\rangle = -1\}$ defined by the inequalities

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It is parallel to the ith facet $\{\langle \rho_i, e\rangle = 0\}$ of the cone $\sigma^\vee$.
- The *Demazure roots* belonging to ρ_i are the lattice vectors from $\mathcal{S}_i$.
- The $\mathbb{G}_a$-subgroup $U_e = \exp(\Bbbk\partial_{\rho_i,e})$ is called the *root subgroup* associated with a Demazure root $e \in \mathcal{S}_i$. For $e, e' \in \mathcal{S}_i$ the root subgroups U_e and $U_{e'}$ commute.

THEOREM (Arzhantsev-Z '20) *Let X be a toric affine variety with no torus factor, and let a subgroup G of $\mathrm{Aut}(X)$ be generated by root subgroups $U_1, \ldots, U_s$. Then either G is a unipotent algebraic group, or G contains a free subgroup of rank two.*

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It has polynomial growth in the former case and exponential in the latter case. If it is highly transitive, then it contains F_2.

PROPOSITION 1 *Consider the group $H = \langle U_1, U_2 \rangle$ generated by the root subgroups $U_i = \exp(t\partial_i)$, $i = 1, 2$, associated with two different ray generators, say, ρ_1 and ρ_2, respectively. Then either H is a unipotent algebraic group, or the subgroup $\langle u_1, u_2 \rangle$ is a free group of rank two for a very general pair $(u_1, u_2) \in U_1 \times U_2$.*

With our choice of X, the Cox ring of $\mathcal{O}(X)$ is the polynomial ring in k variables. This allows to reduce to the setting where $X = \mathbb{A}^k$ and

$$u_1 = (x_1 + sx^c N_1, x_2, \ldots, x_k), \quad u_2 = (x_1, x_2 + tx^d N_2, x_3, \ldots, x_k),$$

with $s, t \in \Bbbk$ and monomials $N_1, N_2 \in \Bbbk[x_3, \ldots, x_k]$.

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with $s, t \in \Bbbk$ and monomials $N_1, N_2 \in \Bbbk[x_3, \dots, x_k]$.
By the Jung-van der Kulk Theorem,

$$\mathrm{GL}_2(K) = \mathrm{Aff}_2(K) *_C \mathrm{Jonq}(K),$$

where

- $K = \Bbbk(s, t, x_3, \dots, x_k)$;
- $\mathrm{Jonq}(K)$ is the de Jonqières triangular subgroup;
- $C = \mathrm{Aff}_2(K) \cap \mathrm{Jonq}(K)$.

- If $c \geq 2$ and $d \geq 2$ then $H = U_1 * U_2$ and $\langle u_1, u_2 \rangle = F_2$ for any pair of nonunit elements (u_1, u_2).

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- If $c = d = 1$ then for a suitable (u_1, u_2), the group $\langle u_1, u_2 \rangle$ surjects onto $\mathrm{SL}_2(\mathbb{Z})$ and so, contains a free subgroup of rank two.

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- If

$$(*) \qquad \min\{c, d\} = 0$$

then H is a unipotent algebraic group.

Let as before

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PROPOSITION 2 *Suppose* $(*)$ *holds for any* $e_i \in R_i, e_j \in R_j$, $i \neq j$, *that is,*

$$\min\{\langle \rho_i, e_j \rangle, \langle \rho_j, e_i \rangle\} = 0.$$

Then G *is a unipotent algebraic group.*

DEFINITION A finite sequence of root derivations

$$\mathcal{D} = (D_1, \dots, D_t, D_{t+1}) \text{ where } D_i = \partial_{\rho_{j(i)}, e_{j(i),i}} \in L_{j(i)}$$

forms a *cycle* if $D_{t+1} = D_1$ and

$$\langle \rho_{j(i+1)}, e_{j(i),i} \rangle > 0 \ \forall i = 1, \dots, t.$$

If this inequality holds and $\rho_{j(t+1)} = \rho_{j(1)}$ but possibly $D_{t+1} \neq D_1$, we say that $\mathcal{D}$ is a *pseudo-cycle*.

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LEMMA *TFAE:*

- L *contains no pseudo-cycle;*
- L *contains no cycle;*
- L *contains no* 2*-cycle;*
- $(*)$ *holds* $\forall e_i \in R_i, e_j \in R_j$, $i \neq j$.

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Suppose $(*)$ holds. Consider the abelian Lie subalgebras of L,

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Due to $(*)$, for any $i \neq j$ there is an alternative:

- either $\langle \rho_i, e_j \rangle = 0 = \langle \rho_j, e_i \rangle \ \forall e_i \in R_i$, $\forall e_j \in R_j$,
- or, up to a transposition, $\exists e_i \in R_i \colon \langle \rho_j, e_i \rangle > 0$, and so, $\langle \rho_i, e_j \rangle = 0 \ \forall \, e_j \in R_j$.

In the first case $[L_i, L_j] = 0$**, and in the second** $0 \neq [L_i, L_j] \subset L_i$**. Anyway, we have** $L = \bigoplus_{i=1}^r L_i$ **and**

$$\dim(L) = \sum_{i=1}^{r} \dim(L_i) = \sum_{i=1}^{r} \mathrm{card}\,(R_i) = \mathrm{card}\,(R).$$

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Let Γ_k be the directed graph on the vertices $L_1, \ldots, L_k$ with edges $[L_j, L_i]$ oriented as follows:

$$[L_j \to L_i] \text{ iff } 0 \neq [L_i, L_j] \subset L_i.$$

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Due to Lemma, L contains no pseudo-cycle. This means that Γ_k is acyclic, that is, has no oriented cycle. Then any connected component of Γ_k has a sink. We may assume L_1 to be a sink.

Deleting this sink L_1 and all the incident edges yields again an acyclic directed graph Γ_{k-1}, which in turn has a sink, which we take for L_2.

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$[L_i, L_1] \subset L_1, \quad i = 2, \ldots, r,$

$[L_i, L_2] \subset L_2, \quad i = 3, \ldots, r,$

$\ldots$

$[L_r, L_{r-1}] \subset L_{r-1}.$

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We show that

- $\dim(L_i) < +\infty \ \forall i = 1, \dots, k;$
- for $N \gg 1$, $\mathrm{ad}(L_j)^N(L_i) = 0 \ \forall j \geq i$, and so,
- $\mathrm{ad}(L)^{Nk}(L) = 0.$

It follows that L is nilpotent and finite-dimensional.