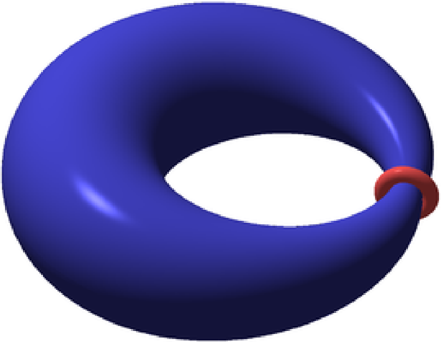
Universal representatives of the homology of algebraic hypersurfaces

Conférence ALKAGE

9-13 march 2020



Damien Gayet (Institut Fourier, Grenoble)

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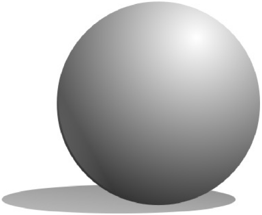
- is generically an orientable compact smooth Riemann surface ;
- connected ;
- with a constant genus $\frac{1}{2}(d-1)(d-2)$.



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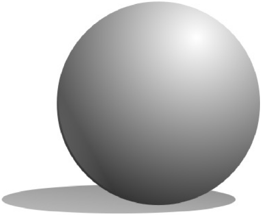 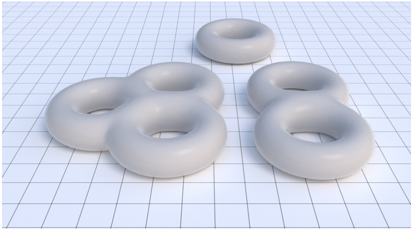

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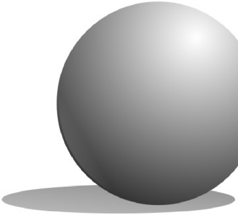

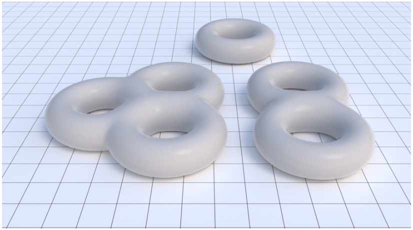

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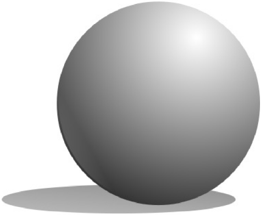

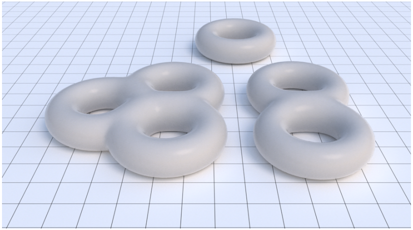

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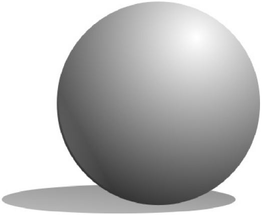

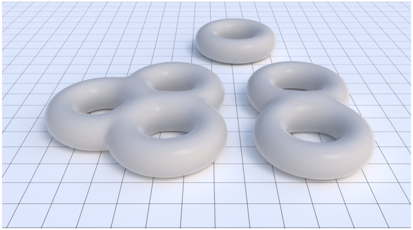

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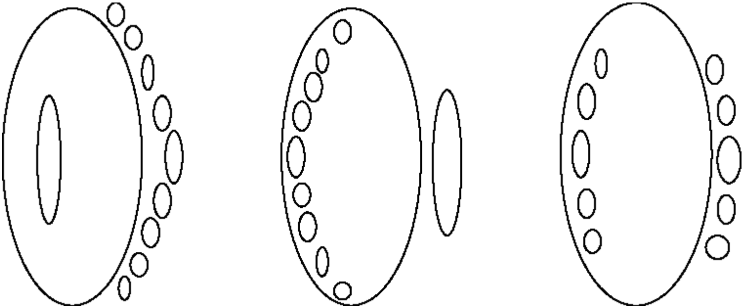
- Same for the moduli space of projective curves

Very different in the real case : various number of components...



... and various possible configurations :
16th Hilbert problem
(here the maximal degree 6 possible curves)

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  1. if P is close to Z_0^d, Z is concentrated near a round sphere,
  2. if P is of high degree d and close to the product of equidistributed d lines, then Z is equidistributed.

Random projective curves

If P is taken at random, what can be said more ?

Theorem (B. Shiffman-S. Zelditch 1998) Almost surely, a sequence of increasing degree random complex curves gets equidistributed in $\mathbb{C}P^2$.

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- This is the Gaussian measure associated to the Fubini-Study L^2-scalar product on the space of polynomials :

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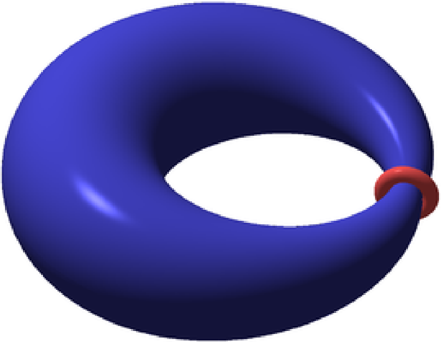
- Generalizes for random sections of high powers of an ample line bundle over a compact Kähler manifold.



What about the length of the **systole** of the random complex curve : its shortest non-contractible real loop ?

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Theorem (M. Mirzakhani 2013). There exist $0 < c < 1$ such that for all $g \geq 2$,

$$c \leq \mathrm{Prob}_{WP}\Big[\text{Length of the systole } \leq 1\Big] \leq 1 - c.$$

Random projective curves

Theorem 1. There exists $c > 0$,

$$\forall d \gg 1,\ c \leq \mathrm{Prob}_{FS}\big[\mathrm{Length}_{\sqrt{d}g_{FS}} \text{ of the systole } \leq 1\big].$$

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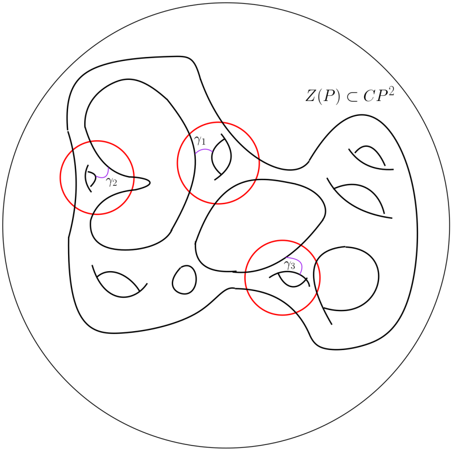

For every d, there exists a basis of $H_1(Z)$ such that a uniform proportion of its elements are represented by small loops with uniform probability

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- complex curves become complex hypersurfaces ;
- non-contractible loops become Lagrangian submanifolds ;
- the useless deterministic bound becomes an non-trivial estimate for homological (Lagrangian) representatives.

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- with a constant diffeomorphism type.

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- $\Rightarrow$ For $n = 3$, $Z \subset \mathbb{C}P^3$ is a connected and simply connected complex surface and its interesting homology lies in $H_2(Z)$, that is for real surfaces inside it.

Small non-trivial submanifolds

Definition. Let (M^n, g) be a compact smooth Riemannian n-manifold. For any $k \in \{1, \cdots, n\}$, let

$$\mathrm{sys}_k(M) := 2 \inf \left\{ \mathrm{diam}\mathcal{L} \mid [\mathcal{L}] \neq 0 \text{ in } H_k(M) \right\}$$

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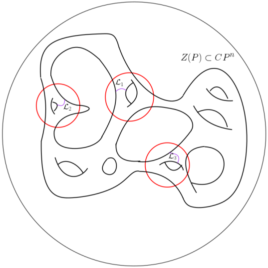
be the Berger k-systole. Facts :

1. Length(systole(M)) $\leq \mathrm{sys}_1(M)$.
2. If $H_k(M) \neq 0$, then $\mathrm{sys}_k(M) > 0$.

Theorem 2 Assume that n is odd. Then,

$$\exists c > 0,\ \forall d \gg 1,\ c \leq \mathrm{Prob}\Big[\mathrm{sys}_{n-1}(Z(P)) \leq 1.\Big]$$



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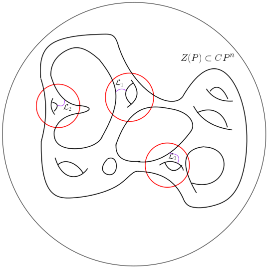

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Recall : $\dim H_*(Z(P)) \sim_{d \to \infty} \dim H_{n-1}(Z(P)) \sim d^n$.

Deterministic corollary Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be *any* compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

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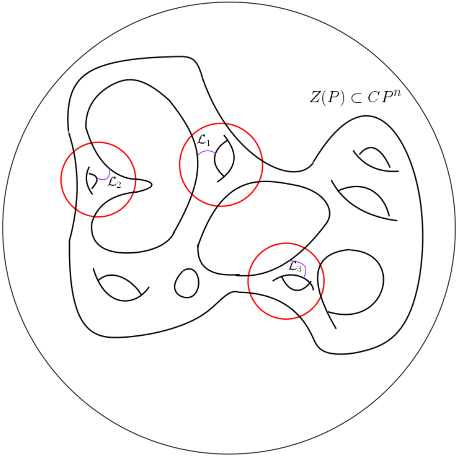
Universal phenomenon : Same holds for zeros of sections of high powers of an ample line bundle over a compact Kähler manifold.



For any real hypersurface $\mathcal{L}$ with non-vanishing Euler characteristic and every large enough degree, there exists a basis of $H_{n-1}(Z)$ such that a uniform proportion of its elements are represented by submanifolds diffeomorphic to $\mathcal{L}$.

Hypersurfaces as symplectic manifolds

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- Hence, if we prove that a property of symplectic nature is true with positive probability, then it is true for *any* hypersurface.

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- The cotangent bundle T^*M of a manifold is naturally symplectic.

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- Easy : the only orientable compact Lagrangian in $(\mathbb{C}^2, \omega_0)$ is the $2-$torus.

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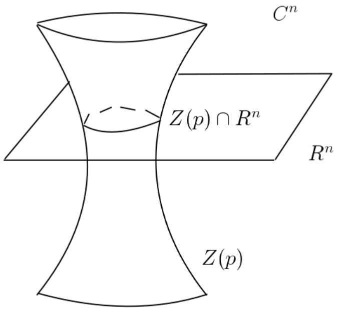
- Motto : Lagrangians are the fundamental bricks of symplectic manifolds and their invariants (in particular with Floer homology).



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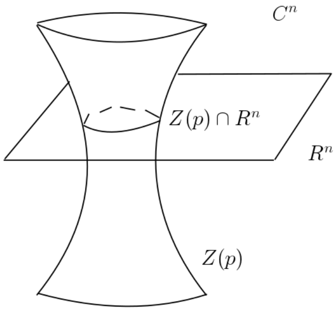

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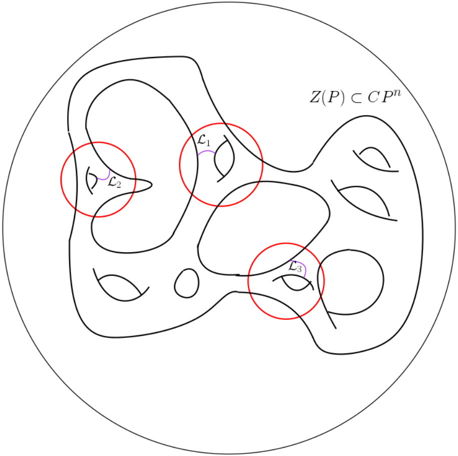

For any real hypersurface $\mathcal{L}$ with non-vanishing Euler characteristic and every large enough degree, there exists a basis of $H_{n-1}(Z)$ such that a uniform proportion of its elements are represented by Lagrangian submanifolds diffeomorphic to $\mathcal{L}$.

Former results

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From tropical arguments :
Theorem (G. Mikhalkin 2004). There exists cd^n disjoint Lagrangian spheres and cd^n Lagrangian tori, whose classes in $H_{n-1}(Z(P))$ are independent, with c explicit and natural.

From random real algebraic geometry :

Theorem (with J.-Y. Welschinger 2014). Let $\mathcal{L} \subset \mathbb{R}^n$ as before. Then there exists (an ugly but explicit and universal) $c > 0$, such that for $d \gg 1$,

$$c < \mathrm{Prob}_{FS,\mathbb{R}}\big[\exists \text{ at least } c\sqrt{d}^n \text{ components of } Z(P) \cap \mathbb{R}P^n \text{ diffeomorphic to } \mathcal{L}\big].$$

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Corollary. At least $c\sqrt{d}^n$ disjoint Lagrangians diffeomorphic to $\mathcal{L}$ in any $Z(P)$.

Proof of Theorem 1 (systoles)

Theorem 1. There exists $c > 0$,

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$$\mathrm{Length}(\gamma) \leq \frac{1}{\sqrt{d}},$$

$$\gamma \text{ non contractible}\Big].$$

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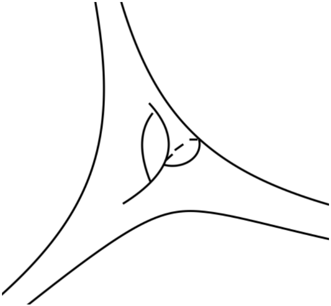
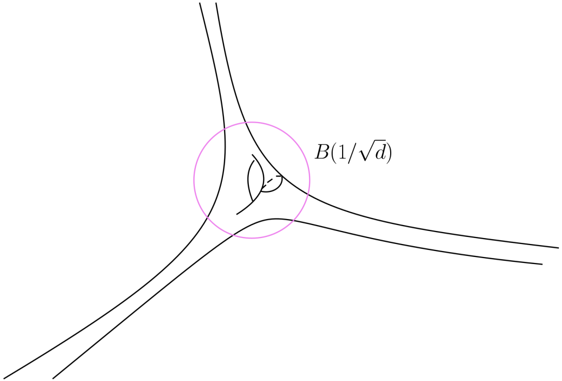


$$Z\Big[Q(1, z_1, z_2)\Big]$$

Rescaling



$$Z\Big[Q(1, \sqrt{d}z_1, \sqrt{d}z_2)\Big]$$

Homogenization

If $Q_d := Z_0^d Q\Big(1, \sqrt{d}(\frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0})\Big)$, then

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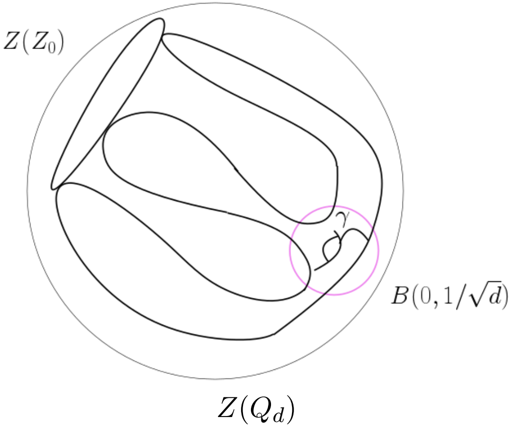

Barrier method

The random P writes

$$\begin{aligned} P &= aQ_d + R, \\ \text{with } a \sim N_{\mathbb{C}}(0,1) \quad &\text{and} \quad R \in Q_d^{\perp} \text{ random independent} \end{aligned}$$

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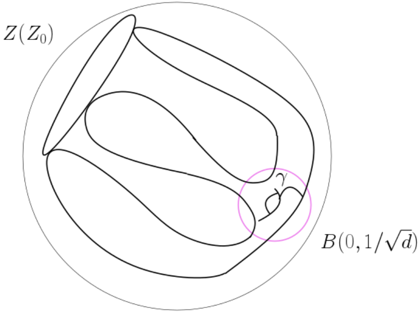

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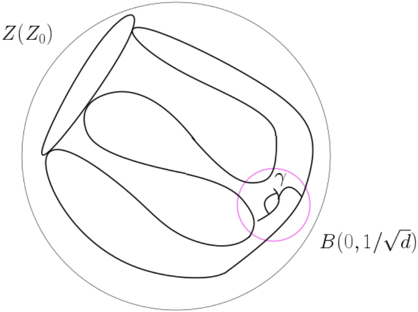

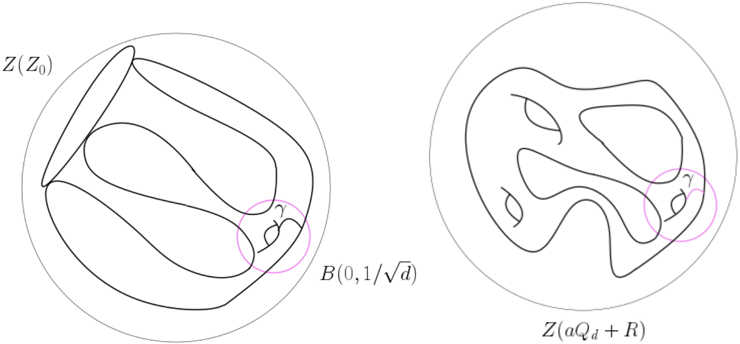

Proposition. With uniform probability in d, R does not destroy the toric shape of $Z(Q_d)$ in $B(x, 1/\sqrt{d})$.

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- Everything is asymptotically independent of d;

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- We can perturb q by random r on the unit ball keeping safe the topology of $Z(aq + r)$.

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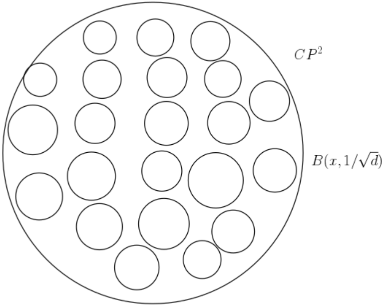
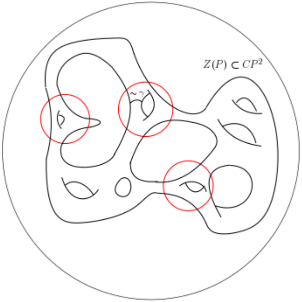
- Hence the Proposition.



There is at least $\sim d^2$ disjoint small balls



With uniform probability, a uniform proportion of these d^2 balls contain the affine torus

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- This means that $1/\sqrt{d}$ is the natural scale of the geometry of degree d algebraic hypersurfaces.
- Universal semi-classical phenomenon : same for sections of an holomorphic line bundles over a complex projective manifold. Reason : universality of peak sections or universal asymptotic behavior of the Bergmann kernel.

Ideas of the proof of Theorem 2

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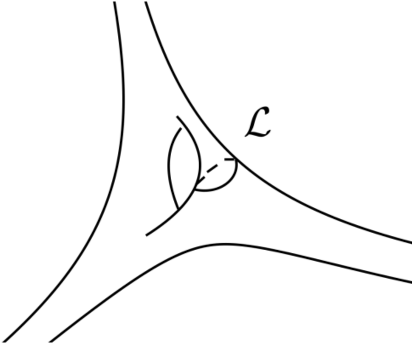
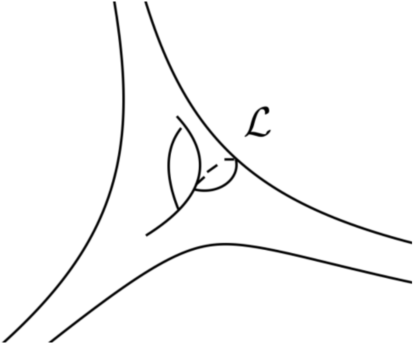
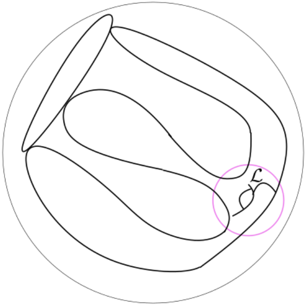
Theorem (Alexander 1936). Every compact smooth real hypersurface $\mathcal{L}$ in $\mathbb{R}^n$ can be C^1-perturbed into a component $\mathcal{L}'$ of an algebraic hypersurface.



- Choose q such that $\mathcal{L} \subset Z(q)$;





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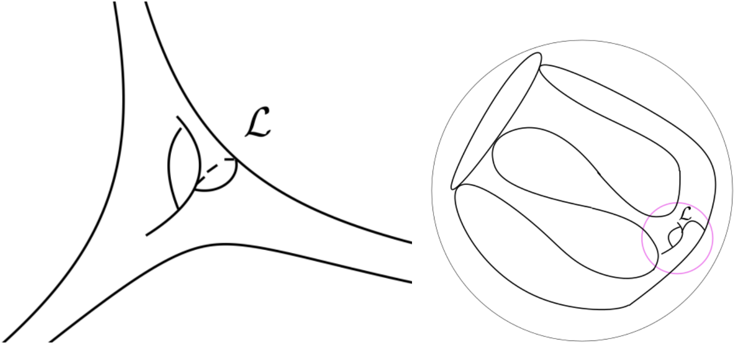
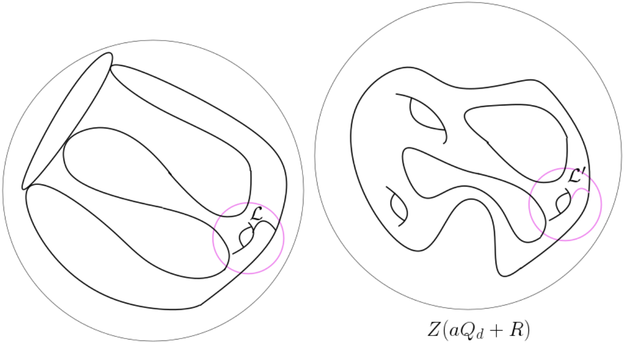


- Choose q such that $\mathcal{L} \subset Z(q)$;
- homogeneize and rescale q into Q_d ;
- decompose $P = aQ_d + R$.



$Z(aQ_d + R)$

Proposition. With uniform probability, in $B(1/\sqrt{d})$,

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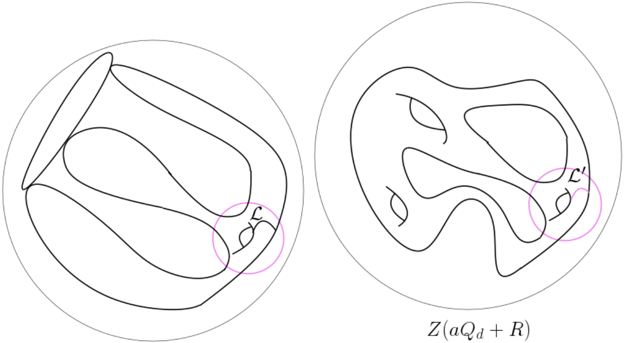

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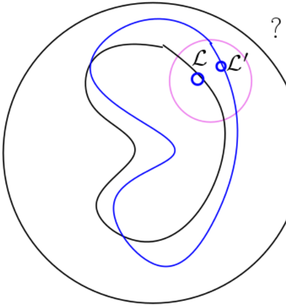
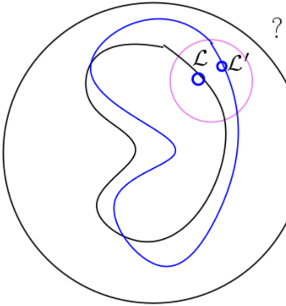
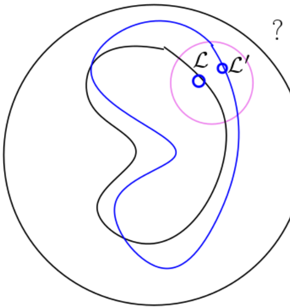
- there exists $\mathcal{L}' \subset Z(P)$ Lagrangian for ω_{FS}.



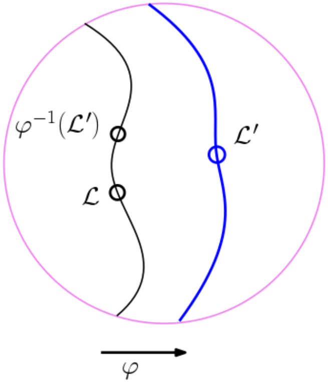
- $\mathcal{L} \subset Z(Q_d)$

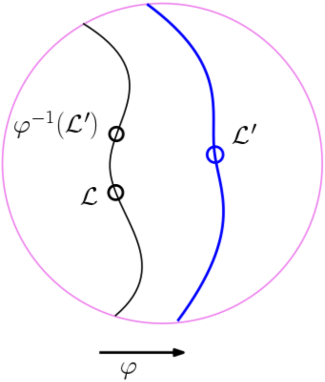


- $\mathcal{L} \subset Z(Q_d)$is Lagrangian for ω_0

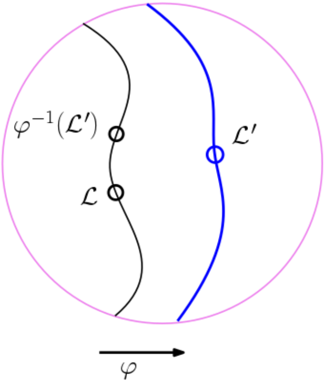


- $\mathcal{L} \subset Z(Q_d)$is Lagrangian for ω_0 ;
- how to find $\mathcal{L}' \subset Z(P)$ Lagrangian for ω_{FS} ?


$\varphi^{-1}(\mathcal{L}')$
$\mathcal{L}'$
$\mathcal{L}$
φ

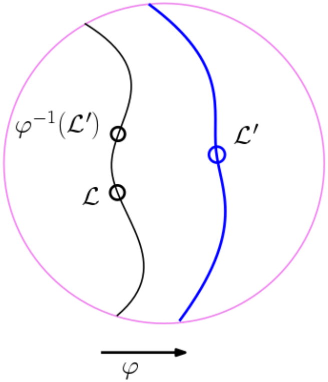


Facts :



Facts :

- $\exists\varphi,\ \varphi(Z(Q_d)) = Z(P)$.



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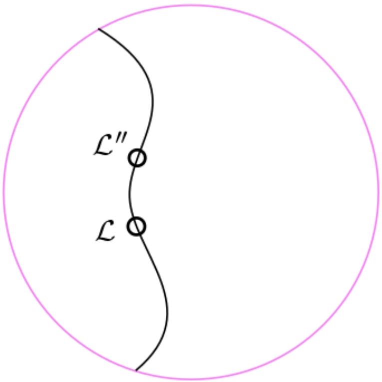
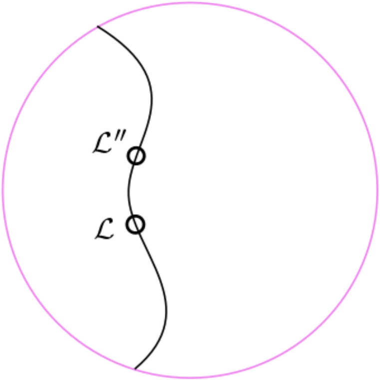
- $\exists \varphi,\ \varphi(Z(Q_d)) = Z(P)$.
- Then

$$\mathcal{L}' \quad \text{Lagrangian for } \omega_{FS} \quad \text{in Z(P)}$$
$$\Leftrightarrow$$
$$\varphi^{-1}(\mathcal{L}') \quad \text{Lagrangian for } \varphi^*\omega_{FS} \quad \text{in } Z(Q_d)$$



- $\mathcal{L}$ Lagrangian for ω_0 in $Z(Q_d)$;



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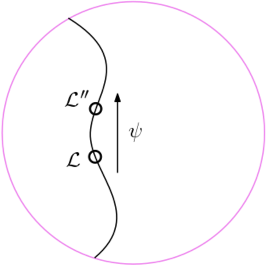
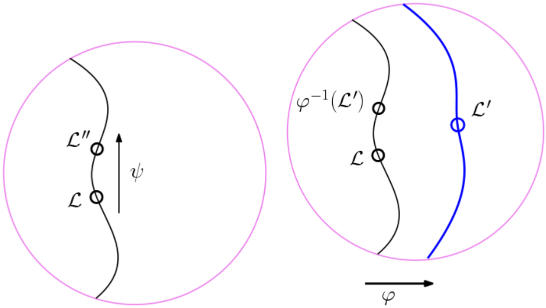


Moser Trick. Let ω symplectic and exact over $Z \cap \mathbb{B}$. Then, there exists $\psi : Z \cap \mathbb{B} \to Z$ such that $\psi^* \omega = \omega_0$.



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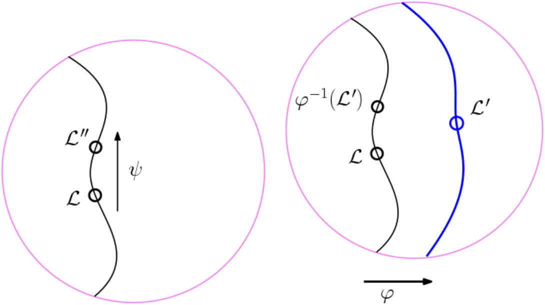

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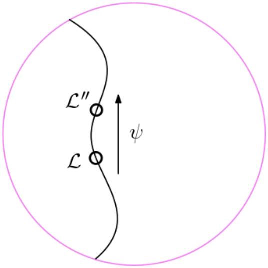
- $\mathcal{L}' = \phi \circ \psi(\mathcal{L})$ is Lagrangian for ω_{FS}

Objection ! It could happen that ψ or φ sends $\mathcal{L}''$ out of the ball !



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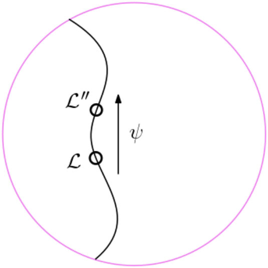

Quantitative Moser Trick. Let ω symplectic and exact over $Z \cap \mathbb{B}$. Then, there exists $\psi : Z \cap \mathbb{B} \to Z$ such that

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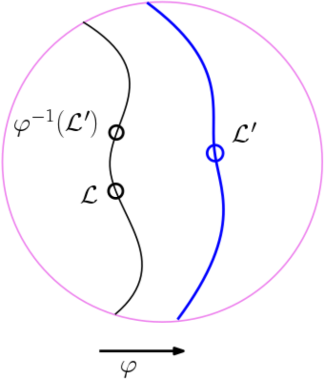
- $|\psi - id|$ is controlled by $|\omega - \omega_0|$



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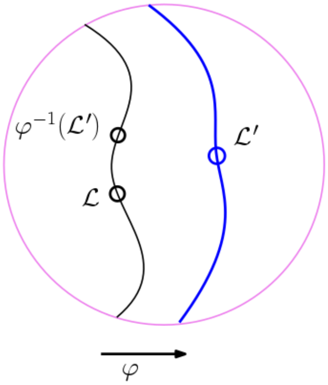

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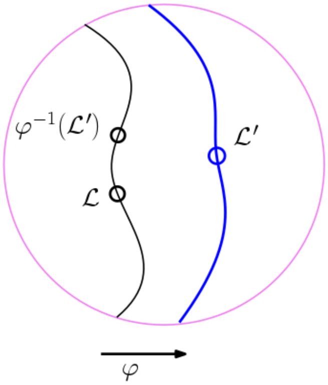

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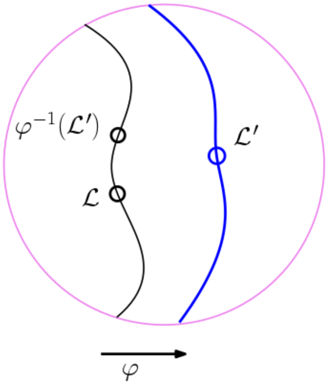

Since

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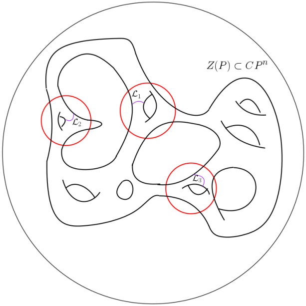
- so that $\mathcal{L}''$ and $\mathcal{L}'$ stay in the ball. $\square$

From one to a lot of Lagrangians



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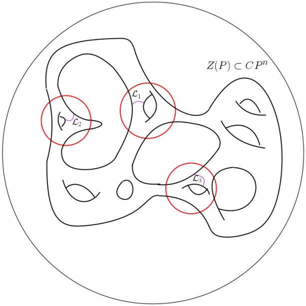

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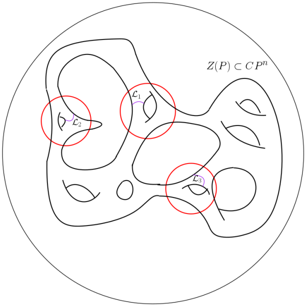

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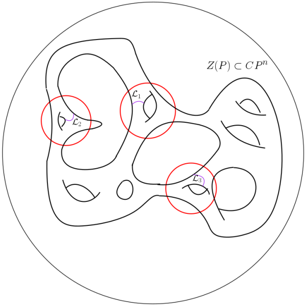

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- With uniform probability, a uniform proportion of them contains a Lagrangian copy of $\mathcal{L}$
- Deterministic conclusion : there exists at least one such hypersurface
- Hence, all of them have cd^n such Lagrangians.

$\square$

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Fact : If $\mathcal{L} \subset (Z, \omega, J)$ is Lagrangian, then

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Corollary The only orientable compact Lagrangian in $\mathbb{R}^4$ is the torus.

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Since ω_t is non-degenerate, this has a solution $(X_t)_t$. $\square$