Universal representatives of the homology of algebraic hypersurfaces

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Topology of planar projective curves

Let $P \in \mathbb{C}_d^{\text{hom}}[Z_0, Z_1, Z_2]$. 
Topology of planar projective curves

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$$Z(P) = \{P = 0\} \subset \mathbb{C}P^2$$
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- is generically an orientable compact smooth Riemann surface;
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- is generically an orientable compact smooth Riemann surface;
- connected;
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- is generically an orientable compact smooth Riemann surface;
- connected;
- with a constant genus \( \frac{1}{2}(d - 1)(d - 2) \).
$d = 1$ or $d = 2$ : sphere

$\dim \text{C}^{\text{hom}}_d[Z_0, Z_1, Z_2] \sim d_g$.

Same for the moduli space of projective curves.
\[ d = 1 \text{ or } d = 2 : \text{sphere} \]
\[ d = 3 : \text{torus} \]
- $d = 1$ or $d = 2$: sphere
- $d = 3$: torus
- $d = 4$: genus $g = 3$
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- $\dim \mathbb{C}_{d}^{\text{hom}}[Z_0, Z_1, Z_2] \sim_d g$. 
- $d = 1$ or $d = 2$ : sphere
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- $d = 4$ : genus $g = 3$
- $\dim \mathbb{C}_{\text{hom}}^d[Z_0, Z_1, Z_2] \sim_d g$
- Same for the moduli space of projective curves
Very different in the real case: various number of components...
... and various possible configurations:
16th Hilbert problem
(here the maximal degree 6 possible curves)
Geometry of planar projective curves

What about the geometry if $Z(P)$ is equipped with the restriction of the ambient metric $g_{FS}$?
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- **W. Wirtinger theorem**: \( \forall P, \text{Vol}(Z(P)) = d. \)
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- **W. Wirtinger theorem**: $\forall P, \text{Vol}(Z(P)) = d$.
- However $Z$ can have very different shapes:
  1. If $P$ is close to $Z_0^d$, $Z$ is concentrated near a round sphere,
  2. If $P$ is of high degree $d$ and close to the product of equidistributed $d$ lines, then $Z$ is equidistributed.
If $P$ is taken at random, what can be said more?

**Theorem (B. Shiffman-S. Zelditch 1998)** Almost surely, a sequence of increasing degree random complex curves gets equidistributed in $\mathbb{C}P^2$. 
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\[ P = \sum_{i_0 + i_1 + i_2 = d} a_{i_0 i_1 i_2} \frac{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2}}{\sqrt{i_0! i_1! i_2!}}, \]

where \( a_{i_0 i_1 i_2} \) are i.i.d. normal variables \( \sim N_{\mathbb{C}}(0, 1) \).
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This is the Gaussian measure associated to the Fubini-Study \( L^2 \)-scalar product on the space of polynomials:

\[ \langle P, Q \rangle_{FS} = \int_{\mathbb{C}P^n} \frac{P(Z)\overline{Q(Z)}}{\|Z\|^{2d}} dvol_{FS}. \]
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Generalizes for random sections of high powers of an ample line bundle over a compact Kähler manifold.
What about the length of the **systole** of the random complex curve: its shortest non-contractible real loop?
The origins: hyperbolic surfaces

Let

\[ \mathcal{M}_g = \{ \text{genus } g \text{ compact smooth surface with a metric of curvature } -1 \}. \]
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- No bound for the diameters, even at fixed $g$. 

Theorem (M. Mirzakhani 2013). There exist $0 < c < 1$ such that for all $g \geq 2$, $c \leq \text{Prob}_{WP}[\text{Length of the systole} \leq 1] \leq 1 - c$. 

10/53
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Random projective curves

**Theorem 1.** There exists $c > 0$,

$$\forall d \gg 1, \ c \leq \text{Prob}_{FS} [\text{Length}_{\sqrt{dg}_{FS}} \text{ of the systole } \leq 1].$$
Recall that $\dim H_1(Z) = 2g \sim d^2$. 
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**Theorem 1’** There exists \( c > 0 \),

\[
\forall d \gg 1, \ c \leq \text{Prob}_{FS} \left[ \exists \gamma_1, \cdots, \gamma_{cd^2}, \forall i, \text{Length}(\gamma_i) \leq 1 \right. \\
\text{and } [\gamma_1], \cdots, [\gamma_{cd^2}] \\
is an independent family of } H_1(Z(P)).
\]
For every $d$, there exists a basis of $H_1(Z)$ such that a uniform proportion of its elements are represented by small loops with uniform probability.
Very useless *deterministic* Corollary. There exists $c > 0$, such that for *any* genus $g$ surface,

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- non-contractible loops become Lagrangian submanifolds;
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- complex curves become complex hypersurfaces;
- non-contractible loops become Lagrangian submanifolds;
- the useless deterministic bound becomes an non-trivial estimate for homological (Lagrangian) representatives.
Higher dimensions

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Higher dimensions

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is generically a smooth complex hypersurface, or $2n - 2$ real submanifold,
Higher dimensions

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- is generically a smooth complex hypersurface, or $2n - 2$ real submanifold,
- with a constant diffeomorphism type.
Lefschetz theorem

\[ \forall k < n - 1, \ H_k(Z(P)) = H_k(\mathbb{C}P^n). \]
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Same for homotopy groups. In particular, \( Z \) is connected for \( n \geq 2 \) and simply connected for \( n \geq 3 \).
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Chern computation

$$\dim H_{n-1}(Z) \sim d^n.$$
Lefschetz theorem

\[ \forall k < n - 1, \ H_k(Z(P)) = H_k(CP^n). \]

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\[ \dim H_{n-1}(Z) \sim d^n. \]

\[ \Rightarrow \] For \( n = 2 \), \( Z \subset CP^2 \) is a connected complex curve and its interesting topology lies in \( H_1(Z) \), whose dimension grows like \( d^2 \).
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Same for homotopy groups. In particular, \( Z \) is connected for \( n \geq 2 \) and simply connected for \( n \geq 3 \).

Chern computation

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⇒ For \( n = 2 \), \( Z \subset \mathbb{C}P^2 \) is a connected complex curve and its interesting topology lies in \( H_1(Z) \), whose dimension grows like \( d^2 \).

⇒ For \( n = 3 \), \( Z \subset \mathbb{C}P^3 \) is a connected and simply connected complex surface and its interesting homology lies in \( H_2(Z) \), that is for real surfaces inside it.
**Definition.** Let $(M^n, g)$ be a compact smooth Riemannian $n$-manifold. For any $k \in \{1, \cdots, n\}$, let

\[
\text{sys}_k(M) := 2 \inf \{ \text{diam}\mathcal{L} \mid [\mathcal{L}] \neq 0 \text{ in } H_k(M) \}
\]

be the Berger $k$-systole.
Small non-trivial submanifolds

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be the Berger \(k\)-systole. Facts:

1. Length(systole\((M)\)) \(\leq\) \(\text{sys}_1(M)\).
2. If \(H_k(M) \neq 0\), then \(\text{sys}_k(M) > 0\).
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1. \(\text{Length}(\text{systole}(M)) \leq \text{sys}_1(M)\).
2. If \(H_k(M) \neq 0\), then \(\text{sys}_k(M) > 0\).
Theorem 2  Assume that $n$ is odd. Then,

$$\exists c > 0, \forall d \gg 1, \ c \leq \text{Prob}[\text{sys}_{n-1}(Z(P)) \leq 1].$$
**Theorem 2'** Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be any compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

$$\exists c > 0, \forall d \gg 1, \ c \leq \text{Prob}\left[ \exists \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \text{ pairwise disjoint, } \forall i, \mathcal{L}_i \sim_{\text{diff}} \mathcal{L}, \ \text{diam}\mathcal{L}_i \leq 1 \right]$$

and $[\mathcal{L}_1], \cdots, [\mathcal{L}_{cd^n}]$ form an independent family of $H_{n-1}(Z(P))$. 

Recall: $\dim H_{*}(\mathbb{Z}(P)) \sim d \to \infty \dim H_{n-1}(\mathbb{Z}(P))$. 

19/53
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Recall: $\dim H_{\ast}(Z(P)) \sim_{d \to \infty} \dim H_{n-1}(Z(P)) \sim d^n$. 
Deterministic corollary  Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be any compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

$$\exists c > 0, \forall d \gg 1, \forall P \in \mathbb{C}^d_{\text{hom}}, \exists \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \subset Z(P)$$

- pairwise disjoint,
- diffeomorphic to $\mathcal{L}$,
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**Deterministic corollary**  Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be any compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

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**Universal phenomenon**: Same holds for zeros of sections of high powers of an ample line bundle over a compact Kähler manifold.
For any real hypersurface $\mathcal{L}$ with non-vanishing Euler characteristic and every large enough degree, there exists a basis of $H_{n-1}(Z)$ such that a uniform proportion of its elements are represented by submanifolds diffeomorphic to $\mathcal{L}$. 
Hypersurfaces as symplectic manifolds

Recall that $\omega_{FS} = g_{FS}(\cdot, J\cdot)$, where $J$ is the complex structure and $g_{FS}$. 
Hypersurfaces as symplectic manifolds

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Facts:

- $(Z(P), \omega_{FS}|_{Z(P)})$ is a symplectic manifold.
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**Facts:**

- $(Z(P), \omega_{FS}|_{Z(P)})$ is a symplectic manifold.
- If $P, Q$ have the same degree,
  \[(Z(P), \omega_{FS}|_{Z(P)}) \sim_{sympl} (Z(Q), \omega_{FS}|_{Z(Q)}).\]
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- Hence, if we prove that a property of symplectic nature is true with positive probability, then it is true for any hypersurface.
Symplectic manifolds

$(M^{2n}, \omega)$ is a symplectic manifold if $\omega$ is a closed non-degenerate 2-form.
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\((M^{2n}, \omega)\) is a symplectic manifold if \(\omega\) is a closed non-degenerate 2-form.

\(\Rightarrow (\mathbb{R}^{2n}, \omega_0)\) with \(\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i\).
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\((M^{2n}, \omega)\) is a *symplectic manifold* if \(\omega\) is a closed non-degenerate 2-form.

- \((\mathbb{R}^{2n}, \omega_0)\) with \(\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i\).
- Darboux theorem: locally any symplectic manifold is symplectomorphic to \((\mathbb{R}^{2n}, \omega_0)\).
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- A real Riemannian surface $(M, g)$ is symplectic when equipped with its area form $d\text{Vol}_g$. 

### Example

- $(\mathbb{C}P^n, \omega_{\text{FS}})$ is symplectic.
- Any complex hypersurface $Z(P) \subset \mathbb{C}P^n$ is symplectic for the restriction of $\omega_{\text{FS}}$.
- The cotangent bundle $T^*M$ of a manifold is naturally symplectic.
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- Motto: Lagrangians are the fundamental bricks of symplectic manifolds and their invariants (in particular with Floer homology).
If \( p \in \mathbb{R}[z_1, \cdots, z_n] \) then

\[
Z(p) \cap \mathbb{R}^n
\]

is Lagrangian in \((Z(p), \omega_0|_{Z(p)})\).
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If $P \in \mathbb{R}_{hom}^d[Z_0, \cdots, Z_n]$ then

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is Lagrangian in $(Z(P), \omega_{FS}|_{Z(P)})$. 
Recall that for a degree $d$ polynomial $P$, $\dim H^\ast(Z(P)) \sim d \to \infty \dim H^{n-1}(Z(P)) \sim d^n$.

Theorem 2. Let $L \subset \mathbb{R}^n$ be any compact hypersurface with $\chi(L) \neq 0$. Then $\exists c > 0, \forall d \gg 1, \forall P \in C^d_{\text{hom}}, \exists L_1, \ldots, L_{cd^n} \subset Z(P)$ are pairwise disjoint, diffeomorphic to $L$, $[L_1], \ldots, [L_{cd^n}]$ form an independent family of $H^{n-1}(Z(P))$, Lagrangian submanifolds of $(Z(P), \omega_{\text{FS}}|_{Z(P)})$. 

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Recall that for a degree $d$ polynomial $P$,

$$\dim H_*(Z(P)) \sim_{d \to \infty} \dim H_{n-1}(Z(P)) \sim d^n.$$  

**Theorem 2.** Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be any compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

$$\exists c > 0, \forall d \gg 1, \forall P \in \mathbb{C}^d_{\text{hom}}, \exists \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \subset Z(P)$$

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- pairwise disjoint,
- diffeomorphic to $L$,
- $[L_1], \cdots, [L_{cd^n}]$ form an independent family of $H_{n-1}(Z(P))$,
- Lagrangian submanifolds of $(Z(P), \omega_{FS}|_{Z(P)})$. 

For any real hypersurface $\mathcal{L}$ with non-vanishing Euler characteristic and every large enough degree, there exists a basis of $H_{n-1}(Z)$ such that a uniform proportion of its elements are represented by Lagrangian submanifolds diffeomorphic to $\mathcal{L}$. 
Former results

From Picard-Lefschetz theory:

Theorem (S. Chmutov 1982). There exists $\sim \frac{d^n}{\sqrt{d}}$ disjoint Lagrangian spheres in $\mathbb{Z}(P)$.

From tropical arguments:

Theorem (G. Mikhalkin 2004). There exists $c_d n$ disjoint Lagrangian spheres and $c_d n$ Lagrangian tori, whose classes in $H^{n-1}(\mathbb{Z}(P))$ are independent, with $c$ explicit and natural.
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From random real algebraic geometry:

**Theorem (with J.-Y. Welschinger 2014).** Let $\mathcal{L} \subset \mathbb{R}^n$ as before. Then there exists (an ugly but explicit and universal) $c > 0$, such that for $d \gg 1$,

$$c < \text{Prob}_{FS,\mathbb{R}}[\exists \text{ at least } c\sqrt{d^n} \text{ components of } Z(P) \cap \mathbb{RP}^n \text{ diffeomorphic to } \mathcal{L}].$$
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**Corollary.** At least $c\sqrt{d^n}$ disjoint Lagrangians diffeomorphic to $\mathcal{L}$ in any $Z(P)$.
Proof of Theorem 1 (systoles)

**Theorem 1.** There exists $c > 0$,

$$\forall d \gg 1, \ c \leq \text{Prob}_{FS}[\text{Length}_{\sqrt{d}g_{FS}} \text{ of the systole} \leq 1].$$
Theorem 1” There exists $c > 0$, 

\[ \forall x \in \mathbb{C}P^n, \forall d \gg 1, \ c \leq \text{Prob}_{FS} \left[ \exists \, \gamma \subset Z(P) \cap B(x, \frac{1}{\sqrt{d}}) \right. \]

\[ \text{Length}(\gamma) \leq \frac{1}{\sqrt{d}}, \]

\[ \gamma \text{ non contractible} \].
Artificial non-contractible curve

Pick a generic $Q \in \mathbb{R}^3_{\text{hom}}[Z_0, Z_1, Z_2]$. 
Artificial non-contractible curve

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Rescaling

\[ Z \left[ Q(1, \sqrt{d}z_1, \sqrt{d}z_2) \right] \]

\[ B(1/\sqrt{d}) \]
Homogenization

If \( Q_d := Z_0^d Q \left( 1, \sqrt{d} \left( \frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0} \right) \right) \), then
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If $Q_d := Z_0^d Q \left(1, \sqrt{d} \left(\frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0}\right)\right)$, then
Barrier method

The random $P$ writes

$$P = a Q_d + R,$$

with $a \sim N_{\mathbb{C}}(0, 1)$ and $R \in Q_d^\perp$ random independent.
Barrier method

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Barrier method

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$$P = aQ_d + R,$$

with $a \sim N_\mathbb{C}(0, 1)$ and $R \in Q_d^\perp$ random independent.
Proposition. With uniform probability in $d$, $R$ does not destroy the toric shape of $Z(Q_d)$ in $B(x, 1/\sqrt{d})$. 
Indeed, over $B(1/\sqrt{d})$ and after rescaling,
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- $Q_d$ looks like the fixed polynomial

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  $$r : \mathbb{B} \to \mathbb{C};$$
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- $P$ looks like
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- $P$ looks like

$$aq + r : \mathbb{B} \to \mathbb{C}.$$

- Everything is asymptotically independent of $d$;
Hence,

- We can perturb $q$ by random $r$ on the unit ball keeping safe the topology of $Z(aq + r)$. 
Hence,

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- The probability that this happens is positive;
- The probability that $Z(aQ_r + R)$ has the good topology is uniformly positive.
Hence,

- We can perturb $q$ by random $r$ on the unit ball keeping safe the topology of $Z(aq + r)$.
- The probability that this happens is positive;
- The probability that $Z(aQ_r + R)$ has the good topology is uniformly positive.
- Hence the Proposition.
There is at least $\sim d^2$ disjoint small balls
With uniform probability, a uniform proportion of these $d^2$ balls contain the affine torus
Why $1/\sqrt{d}$?

This means that $1/\sqrt{d}$ is the natural scale of the geometry of degree $d$ algebraic hypersurfaces.

Universal semi-classical phenomenon: same for sections of an holomorphic line bundles over a complex projective manifold. Reason: universality of peak sections or universal asymptotic behavior of the Bergmann kernel.
Why $1/\sqrt{d}$?

\[ \| Z_0^d \|_{FS} \left( \left[ 1 : \frac{z}{\sqrt{d}} \right] \right) = \frac{|Z_0^d|}{|Z|^d} = (1 + \frac{|z|^2}{d})^{-d/2} \sim d \ e^{-\frac{1}{2}|z|^2}. \]
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Ideas of the proof of Theorem 2
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**Theorem (Alexander 1936).** Every compact smooth real hypersurface $\mathcal{L}$ in $\mathbb{R}^n$ can be $C^1$-perturbed into a component $\mathcal{L}'$ of an algebraic hypersurface.
Choose $q$ such that $\mathcal{L} \subset Z(q)$;
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homogeneize and rescale $q$ into $Q_d$;
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homogeneize and rescale $q$ into $Q_d$;
decompose $P = aQ_d + R$. 
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- $R$ does not kill the shape of $Z(Q_d)$,
**Proposition.** With uniform probability, in $B(1/\sqrt{d})$,  
- $R$ does not kill the shape of $Z(Q_d)$, 
- there exists $\mathcal{L}' \subset Z(P)$ Lagrangian for $\omega_{FS}$.
\[ \mathcal{L} \subset Z(Q_d) \]
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how to find \( \mathcal{L} \subset Z(P) \) Lagrangian for \( \omega_{FS} \)?
Facts:

- $\exists \phi, \phi(Z(Qd)) = Z(P)$

Then $L'$ Lagrangian for $\omega_{FS}$ in $Z(P)$ $\Leftrightarrow$ $\phi^{-1}(L')$ Lagrangian for $\phi^* \omega_{FS}$ in $Z(Qd)$.
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- Then

\[
\mathcal{L}' \quad \text{Lagrangian for } \omega_{FS} \quad \text{in } Z(P)
\]

\[
\Leftrightarrow
\]

\[
\varphi^{-1}(\mathcal{L}') \quad \text{Lagrangian for } \varphi^*\omega_{FS} \quad \text{in } Z(Q_d)
\]
\[ \mathcal{L} \text{ Lagrangian for } \omega_0 \text{ in } Z(Q_d); \]
- $\mathcal{L}$ Lagrangian for $\omega_0$ in $Z(Q_d)$;
- how to find $\mathcal{L}''$ Lagrangian for $\varphi^*\omega_{FS}$ in $Z(Q_d)$?
Moser Trick. Let $\omega$ symplectic and exact over $Z \cap B$. Then, there exists $\psi : Z \cap B \to Z$ such that $\psi^* \omega = \omega_0$. 
Moser Trick. Let $\omega$ symplectic and exact over $\mathbb{Z} \cap \mathbb{B}$. Then, there exists $\psi : \mathbb{Z} \cap \mathbb{B} \to \mathbb{Z}$ such that $\psi^*\omega = \omega_0$.

For us: $\omega = \phi^*\omega_{FS}$,

- $\mathcal{L}'' = \psi(\mathcal{L})$ is Lagrangian, for $\omega$,
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Objection! It could happen that $\psi$ or $\phi$ sends $\mathcal{L}''$ out of the ball!
Moser Trick. Let $\omega$ symplectic and exact over $Z \cap \mathbb{B}$. Then, there exists $\psi : Z \cap \mathbb{B} \to Z$ such that

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Quantitative Moser Trick. Let $\omega$ symplectic and exact over $\mathbb{Z} \cap \mathbb{B}$. Then, there exists $\psi : \mathbb{Z} \cap \mathbb{B} \to \mathbb{Z}$ such that

1. $\psi^* \omega = \omega_0$
2. $|\psi - id|$ is controlled by $|\omega - \omega_0|$
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- \( \omega_{FS} \) is close to \( \omega_0 \),
- with uniform probability \( R \) is small,
- so that \( \varphi \) close to the identity,
- so that \( \mathcal{L}'' \) and \( \mathcal{L}' \) stay in the ball. □
From one to a lot of Lagrangians

There exists $\sim d^n$ balls of size $1/\sqrt{d}$
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- Deterministic conclusion: there exists at least one such hypersurface
From one to a lot of Lagrangians

- There exists $\sim d^n$ balls of size $1/\sqrt{d}$
- With uniform probability, a uniform proportion of them contains a Lagrangian copy of $\mathcal{L}$
- Deterministic conclusion: there exists at least one such hypersurface
- Hence, all of them have $cd^n$ such Lagrangians.
Why non-vanishing Euler characteristics?

**Fact**: If \( \mathcal{L} \subset (Z, \omega, J) \) is Lagrangian, then

\[
NL \sim TL.
\]

Indeed,
\[
\omega = g(J \cdot J),
\]
so that \(JT_L \perp TL\).

\(\blacksquare\)

**If moreover** \(\chi(L) \neq 0\) **then**

\[
0 \neq [L] \in H^{n-1}(Z).
\]

Indeed for \(L\) orientable,
\[
\chi(L) = \# \{\text{zeros of a tangent vector field}\} = \# \{\text{zeros of a normal vector field}\} = [L] \cdot [L].
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**Corollary** The only orientable compact Lagrangian in \(\mathbb{R}^4\) is the torus.
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Moser Trick. Let $\omega$ symplectic and exact over $Z \cap \mathbb{B}$. Then, there exists $\psi : Z \cap \mathbb{B} \to Z$ such that $\psi^* \omega = \omega_0$. 
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Moser Trick. Let $\omega$ symplectic and exact over $Z \cap B$. Then, there exists $\psi : Z \cap B \to Z$ such that $\psi^* \omega = \omega_0$.

Proof. Let $\omega_t := \omega_0 + t(\omega - \omega_0)$. We search $(\phi_t)_t$, such that

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Assume that $(X_t)_t$ is a generating vector field, that is

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This implies $\phi_t^*(\mathcal{L}_{X_t} \omega_t + \partial_t \omega_t) = 0$, which is true if

$$d(\omega_t(X_t, \cdot)) + \omega - \omega_0,$$

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is true, which is true if

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Since $\omega_t$ is non-degenerate, this has a solution $(X_t)_t$. □