Riemann-Roch-Grothendieck theorem for families of curves with hyperbolic cusps and its applications to moduli spaces

Finski Siarhei
Institut Fourier

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Plan of the talk

1. Riemann-Roch-Grothendieck theorem and curvature theorem
2. Motivation
3. Definition of Quillen metric for surfaces with cusps
4. Curvature theorem for family of curves with cusps
5. Sketch of the proof
6. Applications

Riemann-Roch-Grothendieck theorem and curvature theorem

Family setting

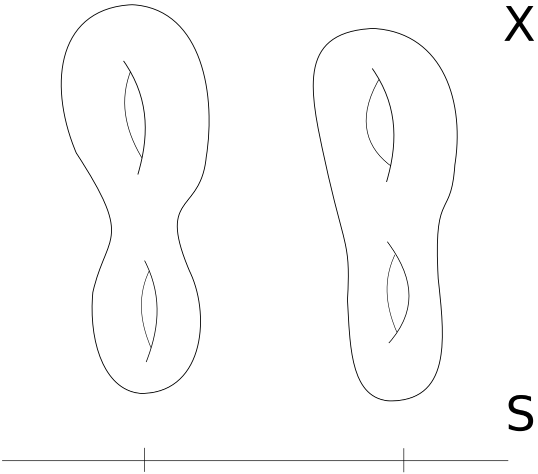
$\pi : X \to S$ proper holomorphic submersion, relative dimension 1

$$\omega_{X/S} = (\Lambda^{\max} T^{*(1,0)} X) \otimes (\Lambda^{\max} T^{*(1,0)} S)^{-1}$$

the relative canonical line bundle of π

$$t \in S, X_t = \pi^{-1}(t)$$

A picture



Dolbeaut complex

ξ a holomorphic vector bundle over X

$$\Omega^{i,j}(X_t, \xi) = \mathscr{C}^\infty(X_t, T^{*(i,j)} X_t \otimes \xi), \quad i, j = 0, 1$$

$$0 \to \Omega^{0,0}(X_t, \xi) \xrightarrow{\overline{\partial}} \Omega^{0,1}(X_t, \xi) \to 0$$

$$H^0(X_t, \xi) = \ker(\overline{\partial}), \qquad H^1(X_t, \xi) = \Omega^{0,1}(X_t, \xi) / \operatorname{Im}(\overline{\partial})$$

The determinant of the cohomology

$$\lambda(\xi)_t = (\Lambda^{\max} H^0(X_t, \xi|_{X_t}))^{-1} \otimes \Lambda^{\max} H^1(X_t, \xi|_{X_t}), \quad t \in S$$

family of complex lines over S

Grothendieck-Knudsen-Mumford :
$\lambda(\xi)_t$, $t \in S$ form a holomorphic line bundle $\lambda(\xi)$ over S

Theorem. (Riemann-Roch-Grothendieck, 1957)

The following identity holds in $H^{\bullet}(S, \mathbb{Q})$:

$$c_1(\lambda(\xi)) = -\int_{\pi} \Big[\mathrm{Td}(\omega_{X/S}) \mathrm{ch}(\xi) \Big]^{[4]}$$

$$\mathrm{Td}(\xi) = 1 + \frac{c_1(\xi)}{2} + \frac{c_1(\xi)^2 + c_2(\xi)}{12} + \dots$$

$$\mathrm{ch}(\xi) = \mathrm{rk}(\xi) + c_1(\xi) + \frac{c_1(\xi)^2 - 2c_2(\xi)}{2} + \dots$$

- Y a complex manifold
  (E, h^E) a holomorphic Hermitian vector bundle over Y
  ∇^E the Chern connection on (E, h^E)
- $R^E = (\nabla^E)^2 \in \Omega^{1,1}(Y, \mathrm{End}(E))$
- $$\mathrm{ch}(E, h^E) = \mathrm{Tr}\Big[\exp\Big(-\frac{R^E}{2\pi\sqrt{-1}}\Big)\Big] \in \oplus_{p\in\mathbb{N}}\Omega^{p,p}(Y)$$
  $$\mathrm{Td}(E, h^E) = \det\Big[\frac{R^E}{\exp(R^E) - 1}\Big] \in \oplus_{p\in\mathbb{N}}\Omega^{p,p}(Y)$$
- $\mathrm{Td}(E, h^E), \mathrm{ch}(E, h^E)$ are closed forms
- Chern-Weil : $\Big[\mathrm{ch}(E, h^E)\Big]_{DR} = \mathrm{ch}(E) \in \oplus_{p\in\mathbb{N}} H^{2p}(Y, \mathbb{R})$
  $\Big[\mathrm{Td}(E, h^E)\Big]_{DR} = \mathrm{Td}(E) \in \oplus_{p\in\mathbb{N}} H^{2p}(Y, \mathbb{R})$

A natural question

$\pi : X \to S$ proper holomorphic submersion, relative dimension 1

$\|\cdot\|_{X/S}^{\omega}$ a Hermitian norm on $\omega_{X/S}$

(ξ, h^{ξ}) a holomorphic Hermitian vector bundle over X

$$\int_{\pi} \Big[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|_{X/S}^{\omega})^2) \mathrm{ch}(\xi, h^{\xi}) \Big]^{[4]}$$

A natural question

$\pi : X \to S$ proper holomorphic submersion, relative dimension 1

$\|\cdot\|^{\omega}_{X/S}$ a Hermitian norm on $\omega_{X/S}$

(ξ, h^{ξ}) a holomorphic Hermitian vector bundle over X

$$c_1(\lambda(\xi), \mathbf{?}) = -\int_{\pi} \Big[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|^{\omega}_{X/S})^2)\mathrm{ch}(\xi, h^{\xi})\Big]^{[4]}$$

- L^2-Hermitian product. Let $\alpha, \alpha' \in \Omega^{0,\bullet}(X_t, \xi)$
  $\langle \alpha, \alpha' \rangle_{L^2} = \int_{X_t} \langle \alpha(x), \alpha'(x) \rangle_h dv_{X_t}(x)$,
  $\langle \cdot, \cdot \rangle_h$ the pointwise Hermitian product induced by $h^\xi, \|\cdot\|^\omega_{X/S}$.

- $0 \to \Omega^{0,0}(X_t, \xi) \xrightarrow{\overline{\partial}} \Omega^{0,1}(X_t, \xi) \to 0$,
  $\square^\xi_t = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$

- $\langle \square^\xi_t \alpha, \alpha \rangle_{L^2} = \langle \overline{\partial}\alpha, \overline{\partial}\alpha \rangle + \langle \overline{\partial}^* \alpha, \overline{\partial}^* \alpha \rangle$,
  $\ker(\square^\xi|_{\Omega^{0,\bullet}(X_t,\xi)}) = \{ s \in \Omega^{0,\bullet}(X_t, \xi) \mid \overline{\partial} s = 0, \ \overline{\partial}^* s = 0 \}$

$$\ker(\square^\xi|_{\Omega^{0,\bullet}(X_t,\xi)}) \to H^\bullet(X_t, \xi) \ \ker(\square^\xi|_{\Omega^{0,\bullet}(X_t,\xi)}) \simeq H^\bullet(X_t, \xi)$$

Hodge theory

- induces the L^2-norm $\|\cdot\|_{L^2}\left(g^{TX_t}, h^\xi\right)$ over
  $\lambda(\xi)_t = (\Lambda^{\max} H^0(X_t, \xi|_{X_t}))^{-1} \otimes \Lambda^{\max} H^1(X_t, \xi|_{X_t})$

From now on $\Box_t^{\xi} := \Box^{\xi}|_{\Omega^{0,0}(X_t,\xi)}$

$\Box_t^{\xi}$ essentially self-adjoint

$\mathrm{Spec}(\Box_t^{\xi}) = \{\lambda_{1,t}, \lambda_{2,t}, \ldots\}$, $\lambda_{i,t}$ non decreasing, $\lambda_{i,t} \to \infty$

$$\det{}'\Box_t^{\xi} = \prod_{\lambda_{i,t} \neq 0}^{\infty} \lambda_{i,t}.$$

Problem : Need to make sense of the infinite product...

Weyl's law : $\lambda_{i,t}$ increase asymptotically linearly with i

$$\zeta_{\xi,t}(s) = \sum_{\lambda_{i,t} \neq 0}^{\infty} \frac{1}{(\lambda_{i,t})^s}, \text{ for } \mathrm{Re}(s) > 1$$

Definition of the determinant. (Ray-Singer, 1973)

$$\det{}' \Box_t^{\xi} = \exp\Big(- \zeta'_{\xi,t}(0) \Big)$$

Quillen norm

Hermitian norm on $\lambda(\xi)$, given by

$$\|\cdot\|^Q \left(g^{TX_t}, h^{\xi}\right) = \left(\det{}' \square_t^{\xi}\right)^{1/2} \cdot \|\cdot\|_{L^2} \left(g^{TX_t}, h^{\xi}\right)$$

Curvature theorem. (Bismut-Gillet-Soulé, 1988)

- Hermitian norm $\|\cdot\|^Q \left(g^{TX_t}, h^{\xi}\right)$ is smooth over S

$$c_1\Big(\lambda(\xi), (\|\cdot\|^Q \left(g^{TX_t}, h^{\xi}\right))^2\Big) = -\int_{\pi} \Big[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|^{\omega}_{X/S})^2)\mathrm{ch}(\xi, h^{\xi})\Big]^{[4]}$$

• Arakelov geometry : arithmetic Riemann-Roch theorem (Gillet-Soulé, Bismut-Lebeau), arithmetic Hilbert-Samuel theorem (Gillet-Soulé, Bismut-Vasserot) 1990's.

• Theory of automorphic forms (Yoshikawa 2004).

• Mirror symmetry (Bershadsky-Cecotti-Ooguri-Vafa '93, Fang-Lu-Yoshikawa '08, Eriksson-Freixas-Mourougane '19).

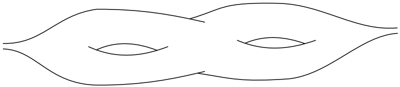
• Explicit evaluation of the values of Selberg zeta function on some modular curves (Freixas 2010, Freixas-v. Pippich 2019).

• Critical phenomenons of some models in statistical mechanics (Duplantier-David 1988, Dubédat 2014, F. 2020).

Motivation

Goal

We want to extend the theory of Quillen metrics to surfaces with hyperbolic cusps and degenerating families with singular fibers



$\overline{M}$ a compact Riemann surface
$D_M = \{P_1, P_2, \dots, P_m\} \subset \overline{M}$, $M = \overline{M} \setminus D_M$

g^{TM} is a Kähler metric on M
$z_1, \dots, z_m$ local holomorphic coordinates, $z_i(0) = \{P_i\}$
Suppose g^{TM} over $\{|z_i| < \epsilon\}$ is induced by

$$\frac{\sqrt{-1}dz_i d\overline{z}_i}{\left|z_i \log |z_i|\right|^2}.$$

We call $(\overline{M}, D_M, g^{TM})$ a surface with cusps

An important example

Suppose $2g(\overline{M}) - 2 + \#D_M > 0$, i.e. $(\overline{M}, D_M)$ is stable

By uniformization theorem, there is exactly one csc -1 complete metric g^{TM}_{hyp} of finite volume on $M = \overline{M} \setminus D_M$

The triple $(\overline{M}, D_M, g^{TM}_{\mathrm{hyp}})$ is a surface with cusps

We want to extend the theory of Quillen metrics to surfaces with hyperbolic cusps and degenerating families with singular fibers

Why ?

- Problem on its own.
- Universal curve $\pi : \mathscr{C}_{g,m} \to \mathscr{M}_{g,m}$ with csc -1 metric $\|\cdot\|_{X/S}^{\omega,\mathrm{hyp}}$

On $\mathscr{M}_{g,m}$, we have $\int_\pi \left[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|_{X/S}^{\omega,\mathrm{hyp}})^2)\right]^{[4]} =^* \omega_{WP}$.

As we expect $c_1(\lambda, (\|\cdot\|^Q)^2) = -\int_\pi \left[\mathrm{Td}(\omega_{X/S}, (\|\cdot\|_{X/S}^{\omega,\mathrm{hyp}})^2)\right]^{[4]}$

$$\text{Regularity of } \|\cdot\|^Q \text{ near } \partial\mathscr{M}_{g,m}$$
$$\Downarrow$$
$$\text{Regularity of } \omega_{WP} \text{ near } \partial\mathscr{M}_{g,m}.$$

- Curvature theorem of Takhtajan-Zograf (csc -1).
- Arithmetic Riemann-Roch theorem for pointed stable curves

Definition of Quillen metric for surfaces with cusps

The L^2-norm

$$\|\cdot\|^Q = \left(\det{}'\Box\right)^{1/2} \cdot \|\cdot\|_{L^2}$$

The L^2-norm

- Let $(\overline{M}, D_M, g^{TM})$ be a surface with cusps
  $\|\cdot\|_M^{\omega}$ the induced Hermitian norm on $\omega_{\overline{M}}$ over M
- $\omega_M(D) = \omega_{\overline{M}} \otimes \mathscr{O}_{\overline{M}}(D_M)$ the twisted canonical line bundle
  $\omega_M(D) \simeq \omega_{\overline{M}},$ over M
  induces the Hermitian norm $\|\cdot\|_M$ on $\omega_M(D)$ over M

  This norm has log singularity $\left\| dz_i \otimes s_{D_M}/z_i \right\|_M = |\log|z_i||$
- (ξ, h^{ξ}) a holomorphic Hermitian vector bundle over $\overline{M}$

$$E_n^{\xi} = \xi \otimes \omega_M(D)^n, \qquad h^{\xi} \otimes (\|\cdot\|_M)^{2n}$$

- For $n \leq 0$, by Hodge theory*
  $\langle \cdot, \cdot \rangle_{L^2}$ induces the L^2-norm $\|\cdot\|_{L^2}$ on
  $\lambda(E_n^{\xi}) = (\Lambda^{\max} H^0(\overline{M}, E_n^{\xi}))^{-1} \otimes \Lambda^{\max} H^1(\overline{M}, E_n^{\xi})$

The determinant

$$\left\|\cdot\right\|^{Q} = \left(\det{}'\Box\right)^{1/2} \cdot \left\|\cdot\right\|_{L^2}$$

Problem with the determinant

$$\square^{E_n^\xi} : \Omega^{0,0}(M, E_n^\xi) \to \Omega^{0,0}(M, E_n^\xi)$$

It is again self-adjoint by the same reason

As M is non-compact, in general $\mathrm{Spec}(\square^{E_n^\xi})$ is not discrete

$$\det{}' \square^{E_n^\xi} \neq \prod_{\lambda_i \neq 0}^{\infty} \lambda_i.$$

1. By relating to a well-defined notion.
2. By analytic methods.
3. By degeneration of the ordinary definition.

Theorem (to be precised later). (-, 2019)

All three approaches give the same result*.

$$\{ \text{ Length of closed geodesics } \} \leftrightarrow \mathrm{Spec}(\Box^{E_n^{\xi}})$$

Suppose (ξ, h^{ξ}) trivial, $(M, D_M, g^{TM}_{\mathrm{hyp}})$ has csc -1
then the set of simple closed geodesics is discrete

$$Z_{(\overline{M}, D_M)}(s) = \prod_{\gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)})$$

γ simple closed geodesics on M; $l(\gamma)$ is the length of γ.

Takhtajan-Zograf definition using Selberg zeta-function, 1991

$$\det{}'_{TZ} \Box^{E_n^\xi} = \begin{cases} Z'_{(\overline{M}, D_M)}(1), & \text{for } n = 0, \\ Z_{(\overline{M}, D_M)}(-n+1), & \text{for } n < 0. \end{cases}$$

Motivated by a theorem of D'Hoker-Phong, 1986, which says that when $m = 0$, two sides of the previous equation coincide*

Limitations of this approach

- Restriction on the topology $2g(\overline{M}) - 2 + \#D_M > 0$.
- Complex structure predefines the Kähler metric.
- No liberty in choosing (ξ, h^ξ).

Analytic approach to the determinant

- $$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} \exp(-\lambda t) t^{s-1} dt$$
- If M is compact, i.e. $m = 0$

$$\zeta_{E_n^\xi}(s) = \sum_{\lambda \in \mathrm{Spec}(\Box^{E_n^\xi}) \setminus \{0\}} \lambda^{-s} \qquad (\star)$$

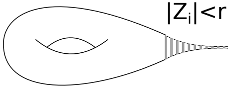
$$= \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}\left[\exp^{\perp}(-t \Box^{E_n^\xi}) \right] t^{s-1} dt \qquad (\star\star)$$

- For $m > 0$?
  Idea : define $\zeta_{E_n^\xi}(s)$ for $m > 0$ using $(\star\star)$ and not $(\star)$
- Problem : $\exp^{\perp}(-t\Box^{E_n^\xi})$ is not of trace class for $m > 0$

- The operator $\exp(-t\square^{E_n^\xi})$ has a smooth Schwartz kernel
$$\exp(-t\square^{E_n^\xi})(x,y) \in (E_n^\xi)_x \otimes (E_n^\xi)^*_y, \qquad x, y \in M$$
$$\exp(-t\square^{E_n^\xi})s = \int_M \Big\langle \exp(-t\square^{E_n^\xi})(x,y), s(y) \Big\rangle dv_M(y).$$

- If $m = 0$, $\mathrm{Tr}\Big[\exp(-t\square^{E_n^\xi})\Big] = \int_{\overline{M}} \mathrm{Tr}\Big[\exp(-t\square^{E_n^\xi})(x,x)\Big] dv_M(x)$.

- Idea : define $\mathrm{Tr}^{\boldsymbol{r}}\Big[\exp(-t\square^{E_n^\xi})\Big]$ by taking the finite part of
$$\int_{M_r} \mathrm{Tr}\Big[\exp(-t\square^{E_n^\xi})(x,x)\Big] dv_M(x)$$
as $r \to 0$, where M_r is the non-striped region



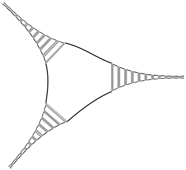
$$P = \mathbb{C}P^1 \setminus \{0, 1, \infty\},$$

g^{TP} hyperbolic metric csc -1 over P

We fix $n \leq 0$

$$g_n(r, t) = \frac{1}{3} \int_{P_r} \exp(-t\Box^{\omega_P(D)^n})(x, x) dv_P(x), \qquad (4.1)$$

where P_r is the non-striped region



Regularizing trace, III

Theorem. (-, 2018)

For any $(\overline{M}, D_M, g^{TM})$, (ξ, h^ξ), $t > 0$, the function

$$\mathbb{R}_{>0} \ni r \mapsto \int_{M_r} \mathrm{Tr}\Big[\exp(-t\Box^{E_n^\xi})(x,x)\Big] dv_M(x) - \mathrm{rk}(\xi) \cdot m \cdot g_n(r,t)$$

extends continuously over $r = 0$.

Regularized heat trace

$$\mathrm{Tr}^{\boldsymbol{r}}\Big[\exp(-t\Box^{E_n^{\xi}})\Big] = \lim_{r\to 0}\bigg(\int_{M_r}\mathrm{Tr}\Big[\exp(-t\Box^{E_n^{\xi}})(x,x)\Big]dv_M(x) - \mathrm{rk}(\xi)\cdot m\cdot g_n(r,t)\bigg).$$

Regularized zeta function

$$\zeta_{E_n^\xi}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \mathrm{Tr}^{\boldsymbol{r}} \Big[\exp^{\perp}(-t\Box^{E_n^\xi}) \Big] t^{s-1} dt.$$

Theorem. (-, 2018)

- $\zeta_{E_n^\xi}(s)$ is well-defined and extends meromorphically to $\mathbb{C}$
- $0 \in \mathbb{C}$ is a holomorphic point of $\zeta_{E_n^\xi}(s)$

Finally, the determinant

Definition of the determinant

$$\det{}' \Box^{E_n^\xi} = \exp\Big(- \zeta'_{E_n^\xi}(0) \Big).$$

Compatibility theorem

Theorem. (-, 2019)

Suppose $(M, D_M, g^{TM}_{\mathrm{hyp}})$ has csc -1, (ξ, h^ξ) trivial. Then for any $m \geq 0$, $n \leq 0$, we have

$$\det{}' \square^{E_n^\xi} =^* \det{}'_{TZ} \square^{E_n^\xi}.$$

$=^*$ means up to some computed universal constant

$m = 0$, D'Hoker-Phong, 1986

Finally, the Quillen norm

Quillen norm

Hermitian norm on $\lambda(E_n^\xi)$, given by

$$\|\cdot\|^Q\left(g^{TM}, h^{E_n^\xi}\right) = \left(\det{}'\square^{E_n^\xi}\right)^{1/2} \cdot \|\cdot\|_{L^2}\left(g^{TM}, h^{E_n^\xi}\right)$$

Curvature theorem for family of curves with cusps

What is a family of curves with cusps ?

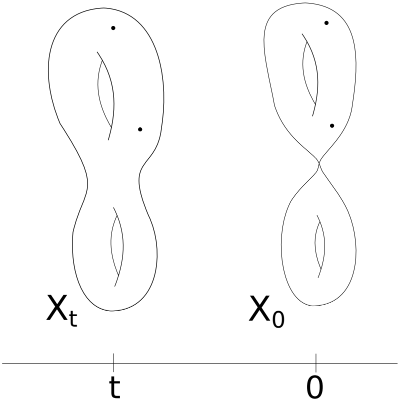
Family of curves with cusps

- $\pi : X \to S$ proper holomorphic of relative dimension 1, $t \in S$, $X_t = \pi^{-1}(t)$ has at most double-point singularities (i.e. those of the form $\{z_0 z_1 = 0\}$)

  $\Sigma_{X/S}$ singular points of the fibers, $\Delta = \pi_*(\Sigma_{X/S})$
- $\sigma_1, \dots, \sigma_m : S \to X \setminus \Sigma_{X/S}$ hol. non intersect. sections $D_{X/S} = \mathrm{Im}(\sigma_1) + \cdots + \mathrm{Im}(\sigma_m)$
- $\|\cdot\|^{\omega}_{X/S}$ Herm. norm on $\omega_{X/S}$ over $X \setminus (|D_{X/S}| \cup \pi^{-1}(|\Delta|))$ $\|\cdot\|^{\omega}_{X/S}|_{X_t}$ induces metric g^{TX_t} on $X_t \setminus |D_{X/S}|$, $t \in S \setminus |\Delta|$ So that $(X_t, \{\sigma_1(t), \dots, \sigma_m(t)\}, g^{TX_t})$ is a surface with cusps

  $(\pi : X \to S, D_{X/S}, \|\cdot\|^{\omega}_{X/S})$ a family of curves with cusps

A picture



$(\pi : X \to S, D_{X/S}, \|\cdot\|^{\omega}_{X/S})$ a family of curves with cusps

$\omega_{X/S}(D) = \omega_{X/S} \otimes \mathscr{O}_X(D_{X/S})$, $\|\cdot\|_{X/S}$

twisted relative canonical line bundle on X

(ξ, h^{ξ}) a holomorphic Hermitian vector bundle over X

$$E_n^{\xi} = \xi \otimes \omega_{X/S}(D)^n$$

$$\lambda(E_n^{\xi})_t = (\Lambda^{\max} H^0(X_t, E_n^{\xi}|_{X_t}))^{-1} \otimes \Lambda^{\max} H^1(X_t, E_n^{\xi}|_{X_t})$$

Grothendieck-Knudsen-Mumford

$\lambda(E_n^{\xi})_t$, $t \in S$ form a holomorphic line bundle $\lambda(E_n^{\xi})$ over S

Quillen norm

We define the Quillen norm on $\lambda(\xi \otimes \omega_{X/S}(D)^n)$ by

$$\|\cdot\|^{Q}\left(g^{TX_t}, h^{\xi} \otimes \|\cdot\|_{X/S}^{2n}\right) = \left(\det{}' \square_t^{E_n^{\xi}}\right)^{1/2} \cdot \|\cdot\|_{L^2}\left(g^{TX_t}, h^{\xi} \otimes \|\cdot\|_{X/S}^{2n}\right).$$

The Wolpert norm

$(\overline{M}, D_M, g^{TM})$, $D_M = \{P_1, \ldots, P_m\}$ surface with cusps
$z_1, \ldots, z_m$ local holomorphic coordinates, $z_i(0) = \{P_i\}$
g^{TM} over $\{|z_i| < \epsilon\}$ is induced by

$$\frac{\sqrt{-1} dz_i d\overline{z}_i}{\left| z_i \log |z_i| \right|^2}$$

Wolpert norm

$\|\cdot\|^W$ on $\otimes_{i=1}^m \omega_{\overline{M}}|_{P_i}$ is defined by

$$\left\| \otimes_i dz_i|_{P_i} \right\|^W = 1.$$

$$on \quad D^* \qquad \frac{\sqrt{-1} dz d\overline{z}}{\left| z \log |z| \right|^2} \quad \rightsquigarrow \quad \left\| dz|_0 \right\|^W = 1 on \quad D^* \qquad \frac{\sqrt{-1} dz d\overline{z}}{\left| z \log |2z| \right|^2}$$

Wolpert norm is related to the “constant term"
of the conformal transformation at cusp

Wolpert norm

We define the Wolpert norm $\|\cdot\|^{W}$ on $\otimes_i \sigma_i^*(\omega_{X/S})$ over S by gluing the Wolpert norms $\|\cdot\|_t^{W}$ on $\otimes_i \omega_{X/S}|_{\sigma_i(t)}$ induced by g^{TX_t}.

We are in the non-compact setting !

We suppose that the metric $\|\cdot\|_{X/S}$ induced on $\omega_{X/S}(D)$ is pre-log-log on X with singularities along $\pi^{-1}(|\Delta|) \cup D_{X/S}$

Notion defied by Burgos Gil-Kramer-Kühn, 2005
It is less restrictive than "good" condition of Mumford, 1977

If $\{z = 0\}$ is a local equation for $\pi^{-1}(|\Delta|) \cup D_{X/S}$ around a smooth point

$$\log(\|v\|_{X/S}) = O((\log|\log|z||)^N)$$

$$\partial \log(\|v\|_{X/S}) = O\Big((\log|\log|z||)^N \frac{dz}{z\log|z|}\Big)$$

$$\partial\bar{\partial} \log(\|v\|_{X/S}) = O\Big((\log|\log|z||)^N \frac{dzd\bar{z}}{|z\log|z||^2}\Big)$$

Wolpert, 1990, (compact case) and Freixas, 2007, (pointed case) proved : the metric of csc -1 on the relative twisted canonical line bundle of universal curve is good

$$\mathscr{L}_n = \lambda(E_n^{\xi})^{12} \otimes (\otimes_i \sigma_i^* \omega_{X/S})^{-\mathrm{rk}(\xi)} \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)} \otimes (\otimes_i \sigma_i^* \det \xi)^6$$

Canonical singular norm

s_Δ the canonical holomorphic section of $\mathscr{O}_S(\Delta)$
$\|\cdot\|_\Delta^{\mathrm{div}}$ on $\mathscr{O}_S(\Delta)$ is defined by $\|s_\Delta\|_\Delta^{\mathrm{div}}(x) = 1, \qquad x \in S \setminus |\Delta|$

$$\|\cdot\|^{\mathscr{L}_n} = \left(\|\cdot\|^Q \left(g^{TX_t}, h^{\xi} \otimes \|\cdot\|_{X/S}^{2n}\right)\right)^{12} \otimes \left(\|\cdot\|^W\right)^{-\mathrm{rk}(\xi)} \otimes \left(\|\cdot\|_\Delta^{\mathrm{div}}\right)^{\mathrm{rk}(\xi)} \otimes \left(\otimes_i \sigma_i^* h^{\det \xi}\right)^3$$

$$\mathscr{L}_n = \lambda(E_n^\xi)^{12} \otimes (\otimes_i \sigma_i^* \omega_{X/S})^{-\mathrm{rk}(\xi)} \otimes \mathscr{O}_S(\Delta)^{\mathrm{rk}(\xi)} \otimes (\otimes_i \sigma_i^* \det \xi)^6$$

$$\|\cdot\|^{\mathscr{L}_n} = \left(\|\cdot\|^Q \left(g^{TX_t}, h^\xi \otimes \|\cdot\|_{X/S}^{2n}\right)\right)^{12} \otimes \left(\|\cdot\|^W\right)^{-\mathrm{rk}(\xi)} \otimes \left(\|\cdot\|_\Delta^{\mathrm{div}}\right)^{\mathrm{rk}(\xi)} \otimes \left(\otimes_i \sigma_i^* h^{\det \xi}\right)^3$$

Theorem. (-, 2018)

$\|\cdot\|^{\mathscr{L}_n}$ extends continuously* over $|\Delta|$, smooth* over $S \setminus |\Delta|$, and on the level of currents over S :

$$c_1\left(\mathscr{L}_n, (\|\cdot\|^{\mathscr{L}_n})^2\right) = -12 \int_\pi \Big[\mathrm{Td}(\omega_{X/S}(D), \|\cdot\|_{X/S}^2) \cdot \mathrm{ch}(\xi, h^\xi)\mathrm{ch}(\omega_{X/S}(D)^{2n}, \|\cdot\|_{X/S}^{2n})\Big]^{[4]}$$

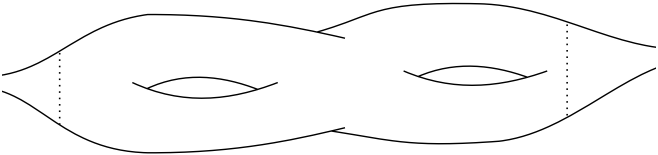
If no cusps, no degeneration of metric, Bismut-Bost, 1990
If csc -1, no v.b., no degeneration, Takhtajan-Zograf, 1991

Sketch of the proof

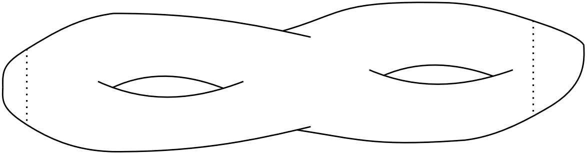
Relative compact perturbation theorem

A notion of flattening

Flattening of a metric with cusps g^{TM}



is a Kähler metric g_{f}^{TM} on $\overline{M}$ such that



The same for $\|\cdot\|_M$

Let $(\overline{M}, D_M, g^{TM})$ be a surface with cusps,

(ξ, h^{ξ}) Hermitian vector bundle over $\overline{M}$

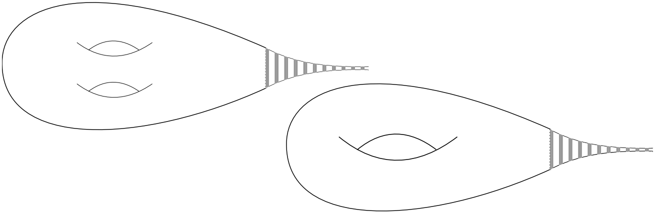
$g_{\mathrm{f}}^{TM}, \|\cdot\|_M^{\mathrm{f}}$ the flattenings of $g^{TM}, \|\cdot\|_M$

We want to understand how to calculate

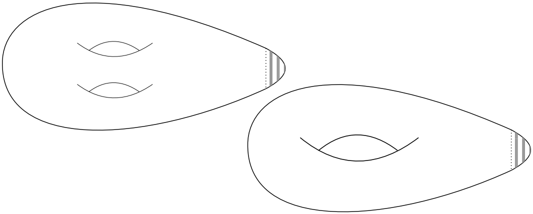
$$\frac{\|\cdot\|_Q \left(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}\right)}{\|\cdot\|_Q \left(g_{\mathrm{f}}^{TM}, h^{\xi} \otimes (\|\cdot\|_M^{\mathrm{f}})^{2n}\right)}$$

In other words :

How Quillen metric changes under compact perturbation ?



Two flattenings g_{f}^{TM}, g_{f}^{TN} of g^{TM}, g^{TN} are called compatible if



$(\overline{M}, D_M, g^{TM})$, $(\overline{N}, D_N, g^{TN})$ surfaces with cusps, $\#D_M = \#D_N$

(ξ, h^ξ) Hermitian vector bundle over $\overline{M}$

$g_{\mathrm{f}}^{TM}, g_{\mathrm{f}}^{TN}, \|\cdot\|_M^{\mathrm{f}}, \|\cdot\|_N^{\mathrm{f}}$ compatible flattenings of $g^{TM}, g^{TN}, \|\cdot\|_M \|\cdot\|_N$

Theorem. (-, 2018)

For simplicity, suppose (ξ, h^ξ) is trivial

$$\frac{\|\cdot\|_Q \left(g^{TM}, h^\xi \otimes \|\cdot\|_M^{2n}\right)}{\|\cdot\|_Q \left(g_{\mathrm{f}}^{TM}, h^\xi \otimes (\|\cdot\|_M^{\mathrm{f}})^{2n}\right)} = \left(\frac{\|\cdot\|_Q \left(g^{TN}, \|\cdot\|_N^{2n}\right)}{\|\cdot\|_Q \left(g_{\mathrm{f}}^{TN}, (\|\cdot\|_N^{\mathrm{f}})^{2n}\right)} \right)^{\mathrm{rk}(\xi)}$$

$(\overline{M}, D_M, g^{TM})$, $(\overline{N}, D_N, g^{TN})$ surfaces with cusps, $\# D_M = \# D_N$

(ξ, h^{ξ}) Hermitian vector bundle over $\overline{M}$

$g_{\mathrm{f}}^{TM}, g_{\mathrm{f}}^{TN}, \|\cdot\|_M^{\mathrm{f}}, \|\cdot\|_N^{\mathrm{f}}$ compatible flattenings of $g^{TM}, g^{TN}, \|\cdot\|_M \|\cdot\|_N$

Theorem. (-, 2018)

$$\frac{\|\cdot\|_Q \left(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}\right)}{\|\cdot\|_Q \left(g_{\mathrm{f}}^{TM}, h^{\xi} \otimes (\|\cdot\|_M^{\mathrm{f}})^{2n}\right)} = \left(\frac{\|\cdot\|_Q \left(g^{TN}, \|\cdot\|_N^{2n}\right)}{\|\cdot\|_Q \left(g_{\mathrm{f}}^{TN}, (\|\cdot\|_N^{\mathrm{f}})^{2n}\right)} \right)^{\mathrm{rk}(\xi)}$$

$$\cdot \exp\left(\frac{1}{2} \int_M c_1(\xi, h^{\xi}) \Big(2n \ln(\|\cdot\|_M^{\mathrm{f}} / \|\cdot\|_M) + \ln(g_{\mathrm{f}}^{TM} / g^{TM}) \Big) \right)$$

Anomaly formula

Anomaly formula, setting

$(\overline{M}, D_M)$ a pointed Riemann surface
g^{TM}, g_0^{TM} metrics with cusps at D_M

$\|\cdot\|_M, \|\cdot\|_M^0$ the norms induced by g^{TM}, g_0^{TM} on $\omega_M(D)$

$\|\cdot\|^W, \|\cdot\|_0^W$ the associated Wolpert norms on $\otimes_{P \in D_M} \omega_{\overline{M}}|_P$

ξ holomorphic vector bundle on $\overline{M}$
h^ξ, h_0^ξ Hermitian metrics on ξ over $\overline{M}$

Anomaly formula

Theorem. (-, 2018)

$$2\log\Big(\|\cdot\|_Q \left(g_0^{TM}, h_0^{\xi} \otimes (\|\cdot\|_M^0)^{2n}\right) \Big/ \|\cdot\|_Q \left(g^{TM}, h^{\xi} \otimes \|\cdot\|_M^{2n}\right) \Big)$$

$$= \int_M \Big[\text{Bott-Chern terms, analogic to the anomaly}$$

$$\text{for compact manifolds of Bismut-Gillet-Soulé}\Big]$$

$$- \frac{\mathrm{rk}(\xi)}{6} \log \left(\|\cdot\|^W / \|\cdot\|_0^W \right) + \sum \log \left(\det(h^{\xi}/h_0^{\xi})|_{P_i} \right).$$

Applications

Theorem of Wolpert

$\overline{\mathscr{M}}_{g,m}$ the Deligne-Mumford compactification of the moduli space of pointed Riemann surfaces $\mathscr{M}_{g,m}$

$\pi : \overline{\mathscr{C}}_{g,m} \to \overline{\mathscr{M}}_{g,m}$ the universal family of pointed curves

$\omega_{g,m}(D)$ the associated twisted canonical line bundle
$\|\cdot\|_{g,m}$ the norm on $\omega_{g,m}(D)$ induced by constant scalar curvature -1 metric on the fiber

ω_{WP} the Weil-Petersson Kähler form on $\mathscr{M}_{g,m}$

Theorem. (Wolpert, 1986)

$$\omega_{WP} = 12\pi^2 \cdot \int_\pi \left[\mathrm{Td}(\omega_{g,m}(D), \|\cdot\|_{g,m}^2) \right]^{[4]}$$

Consequence on the growth of ω_{WP}, I

But then as a current ω_{WP} satisfies

$$\omega_{WP} = -\pi^2 \frac{\partial\overline{\partial} \log(\|s\|^{\mathscr{L}_0})^2}{2\pi\sqrt{-1}}$$

s - local holomorphic section of $\mathscr{L}_0$

Corollary. (-, 2018)

ω_{WP} has local continuous potential over $\overline{\mathscr{M}}_{g,m}$.

Wolpert, 1986 proved it by different methods

Consequence on the growth of ω_{WP}, II

$$\mathscr{L}_0 = \lambda(\mathscr{O}_{\overline{\mathscr{M}}_{g,m}})^{12} \otimes (\otimes_i \sigma_i^* \omega_{g,m})^{-1} \otimes \mathscr{O}_{\overline{\mathscr{M}}_{g,m}}(\partial \mathscr{M}_{g,m})$$

Corollary. (-, 2018)

$\omega_{WP} = \pi^2 \alpha + d\beta$, where $\alpha = -c_1(\mathscr{L}_0, (\|\cdot\|_{sm}^{\mathscr{L}_0})^2)$ for some smooth norm $\|\cdot\|_{sm}^{\mathscr{L}_0}$ on $\mathscr{L}_0$ and β have mild singularities near $\partial \mathscr{M}_{g,m}$. As a trivial consequence, we obtain

$$\int_{\mathscr{M}_{g,m}} \omega_{WP}^{\wedge \max} = (-\pi^2)^{\dim_{\mathbb{C}}(\mathscr{M}_{g,m})} \cdot \int_{\overline{\mathscr{M}}_{g,m}} c_1(\mathscr{L}_0)^{\cap \max}$$

$$\in \pi^{2 \dim_{\mathbb{C}}(\mathscr{M}_{g,m})} \cdot \mathbb{Q}^*$$

Wolpert, 1985 proved the second part by different methods

Thank you !