Non-reductive geometric invariant theory and hyperbolicity

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joint work with Frances Kirwan
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Hyperbolicity: a very brief overview

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X is said to be **weakly hyperbolic** if there exists an algebraic variety $Y \subsetneq X$ such that for all non-constant holomorphic $f : \mathbb{C} \to X$ one has $f(\mathbb{C}) \subset Y$.

Brief overview

Green-Griffiths-Lang conjecture,1981 Every projective algebraic variety X of general type is weakly hyperbolic.

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- McQuillen (1998): Positive answer for surfaces if $c_1^2 - c_2 > 0$.
- Demailly and Siu (90's): Strategy for projective hypersurfaces
- Siu (1996) Positive answer for the GGL conjecture hypersurfaces of high degree.
- Diverio, Merker, Rousseau (2009): Effective lower bound, $deg(X) > 2^{n^5} \Rightarrow GGL$ (degree improved by B., Merker, Demailly, but these are exponential bounds)
- Brotbek (2016) Proof of Kobayashi for sufficiently high degree (not effective). Effective (exponential) bound by Deng (improved by Demailly, Merker)

Today: Report on the proof of polynomial GGL and Kobayashi theorem.

Let $f : \mathbb{C} \to X, \quad t \to f(t) = (f_1(t), f_2(t), \dots, f_n(t))$ be a curve written in some local holomorphic coordinates $(z_1, \dots, z_n)$ on X. The **k-jet bundle** is

$$J_k X = \{f_{[k]} : f : (\mathbb{C}, 0) \to X\} \to X$$

sending $f_{[k]}$ to $f(0)$. Fibre is

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The group of reparametrizations $\mathrm{Diff}_k = J_k^{reg}(1,1)$ acts fiberwise on $J_k X$. $\mathrm{Diff}_k = U_k \rtimes \mathbb{C}^*$, and for $\lambda \in \mathbb{C}^*$

$$(\lambda \cdot f)(t) = f(\lambda \cdot t), \text{ so } \lambda \cdot (f', f'', \ldots f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots \lambda^k f^{(k)}).$$

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Consider **algebraic differential operators = polynomial functions on** J_kX. Locally in multi-index notation

$$Q(f',f'',\ldots,f^{(k)})=\sum_{\alpha_i\in\mathbb{N}^n}a_{\alpha_1,\alpha_2,\ldots\alpha_k}(f(t))(f'(t)^{\alpha_1}f''(t)^{\alpha_2}\cdots f^{(k)}(t)^{\alpha_k},$$

where $a_{\alpha_1,\alpha_2,\ldots\alpha_k}(z)$ are holomorphic coefficients on X and $t\to z=f(t)$ is a curve. Q is homogeneous of weighted degree m under the $\mathbb{C}^*$ action iff

$$Q(\lambda f',\lambda^2f'',\ldots\lambda^kf^{(k)})=\lambda^mQ(f',f'',\ldots f^{(k)}).$$

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(Demailly, '95) The bundle of invariant jet differentials of order k and weighted degree m is the subbundle $E_{k,m} \subset E^{GG}_{k,m}$, whose elements are invariant under arbitrary changes of parametrization, i.e. for $\phi \in \mathrm{Diff}_k$

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More precisely: We want meaningful projective completions of $J_k X/\mathrm{Diff}_k \subset \mathbb{P}^N$ such that (global) sections of $\mathcal{O}_{\mathbb{P}^N}(1)$ are invariant jet differentials. This should be a fibrewise compactification $\mathcal{X}_k \to X$ whose fiber is isomorphic to a projective completion of $J_k(1,n)/\mathrm{Diff}_k$.

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If $f(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0) \in J_k(1,n)$ and $\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_k z^k \in \mathrm{Diff}_k$ then

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$$f\circ\varphi(z) = (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \ldots + \left(\sum_{i_1+\ldots+i_l=k} \frac{f^{(l)}(0)}{l!}\alpha_{i_1}\ldots\alpha_{i_l}\right)z^k =$$

$$= (f'(0), \ldots, f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \ldots & 2\alpha_1\alpha_{k-1}+\ldots \\ 0 & 0 & \alpha_1^3 & \ldots & 3\alpha_1^2\alpha_{k-2}+\ldots \\ 0 & 0 & 0 & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \alpha_1^k \end{pmatrix} = U_k \rtimes \mathbb{C}^*$$

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There are 3 different compactifications of this quasi-projective quotient.

1. The fibre of the Demailly-Semple bundle (tower of projective bundles) is a smooth compactification. This was used for over 30 years to attack the conjectures, and in particular Diverio-Merker-Rousseau in 2009 gave triplke exponential (but effective!) degree bound in the GGL conjecture. We can derive iterated residue formulae for intersection numbers on this using localisation and improve the bound to exponential (Bérczi 2011)

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3. Diff_k is not reductive. For non-reductive H, $\mathcal{O}(X)^H$ is not necessarily finitely generated (Nagata). So $Proj(\mathcal{O}(X)^G)$ is not a projective variety, and the image of $\pi : X \to Proj(\mathcal{O}(X)^G)$ is just constructible.

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But: Between 2010-2017 Kirwan-Bérczi-Doran-Hawes developed non-reductive GIT, and gave a third way to compactify $J_k(1,n)/\mathrm{Diff}_k$. This has symplectic description and hence a powerful cohomology theory. Moreover there is a master blow-up of the NRGIT quotient which maps to both two previous comp's.

Strategy

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Corollary

1. Let σ be a nonzero element of

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   then $f_{[k]}(\mathbb{C}) \subset Z_\sigma$.
2. If σ_j is a set of sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap Z_{P_j})$, hence GGL holds if there are enough independent differential equations so that $Y = \pi_k(\bigcap(Z_{P_j})) \subsetneq X$.

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It is crucial to control in a more precise way the order of vanishing of these differential operators along the ample divisor. One gets

Theorem (Diverio-Merker-Rousseau, 2009, Darondeau 2016) Assume that $n=k$, and there exist a $\delta=\delta(n)>0$ and $D=D(n,\delta)$ such that

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whenever $\deg(X)>D(n,\delta)$ provided that $m>m_{D,\delta,n}$ is large enough. Then GGL holds for

$$\deg(X)\geq\max(D(n,\delta),\frac{5n+3}{\delta}+n+2).$$

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$$h^0(X, L^{\otimes m}) - h^1(X, L^{\otimes m}) \geq \frac{m^n}{n!}(F^n - nF^{n-1}G) + \mathcal{O}(m^{n-1}) > 0$$

In short: Problem is reduced to algebraic geometry and integration on $\mathcal{X}_k$: we need to prove the positivity of a cohomological intersection number on the r.h.s.

Non-reductive GIT: arXiv:1607.04181

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Moreover without condition (*) there is a GIT blow-up process to provide a $\hat{U}$-equivariant blow-up $\tilde{X}$ of X for which (*) holds. Theorem works for graded linear groups in general.

$\hat{U} = U \rtimes \mathbb{C}^*$ positively graded $\mathbb{C}^*$-extension of a unipotent group U. Let X be a projective variety endowed with a linear action of $\hat{U}$ with respect to some ample line bundle L. Assume (s=ss) condition of the $\hat{U}$-Theorem holds (otherwise we blow-up!).

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Note that a key difference to the reductive picture of Kirwan is that for $\hat{U}$ actions we have a distinguished critical set $X_{\mathrm{min}}^{\mathbb{C}^*}$ which carries all cohomological information.

Reductive abelianisation: The following diagram of Shaun Martin relates $X//G$ and $X//T_{\mathbb{C}}$ through a fibering and an inclusion:

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Classical results:

- (SM,'00) $H^*(X//G,\mathbb{Q}) \simeq \frac{H^*(X//T,\mathbb{Q})^W}{ann(e)}$ where $\mathrm{ann}(e) = \{c \in H^*(X//T,\mathbb{Q})^W | c \cup e = 0\}$.

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Non-reductive case:

$$\begin{tikzcd} \mu_K^{-1}(0)/S^1 \arrow[r, hook, "j"] \arrow[d, "\pi"] & \mu_{\hat{U}}^{-1}(0)/S^1 = X//\hat{U} \arrow[r, hook, "i"] & \mu_{S^1}^{-1}(0)/S^1 = X//\mathbb{C}^* \\ \mu_K^{-1}(0)/K = X//G \end{tikzcd} \qquad (2)$$

The description of the corresponding normal bundles is similar using the line bundles corresponding to the roots of U.

Theorem [Non-reductive abelianisation, B-Kirwan '19]
(1) Cohomology

$$H^*(X//\hat{U}, \mathbb{Q}) \simeq H^*(X//\mathbb{C}^*, \mathbb{Q})/ann(e(V_{\mathfrak{u}}))$$

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(3) Integration formula For any Chern polynomial $\phi(V) \in \mathbb{C}[c_1(V), c_2(V), \ldots]$ whose degree is the dimension of $X//\hat{U}$ we have

$$\int_{X//\hat{U}} \phi(V) = n_{\mathbb{C}^*} \mathrm{Res}_{z=\infty} \int_{X_{\min}^{\mathbb{C}^*}} \frac{i^*_{X_{\min}^{\mathbb{C}^*}}(\phi(V) \cup e(V_\mathfrak{u}))dz}{e^T(\mathcal{N}_{X_{\min}^{\mathbb{C}^*}})}$$

Main theorem: arXiv:1909.11417

Polynomial Green-Griffiths-Lang theorem (B-Kirwan '19). Let $X \subseteq \mathbb{P}^{n+1}$ be a generic smooth projective hypersurface of degree $\deg(X) \geq 16n^5(5n+4)$. Then X is weakly hyperbolic.

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$$\pi_*\mathcal{O}_{\mathcal{X}_k}(m) \subseteq \mathcal{O}(E_{k,\leq 2Nkm})$$

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where $h = c_1(\mathcal{O}_X(1)), u = c_1(\mathcal{O}_{\mathcal{X}_k}(1))$. The moment map description gives

$$\int_{\mathcal{X}_k} I_{n,\delta} = \int_X \operatorname*{Res}_{w=\infty} \operatorname*{Res}_{z=\infty} \frac{(n-1)!(z-w)^{n-1} I_{n,\delta}(z,w,h)dwdz}{(-1)^{n-1} z \prod_{l=0}^{n-1}(lz-(l+1)w)^n} \prod_{l=0}^{n-1} \left(1 + \frac{dh}{lz-(l+1)w}\right)\left(1 - \frac{h}{lz-(l+1)w} + \dots\right)^{n+2} > 0$$